ON CERTAIN POLYNOMIAL AND OTHER APPROXIMATIONS TO ANALYTIC FUNCTIONS*

BY

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PART I. INTRODUCTION

1. **The Lagrange interpolation polynomials.** This paper deals largely with certain polynomial approximations to analytic functions of a complex variable, that are somewhat analogous to the Lagrange interpolation polynomials. The latter, it will be recalled, are defined as follows:

Given an analytic function $f(z)$ of the complex variable z, and a set of n points $z=a_1, a_2, \cdots, a_n$, the corresponding Lagrange interpolation polynomial is the polynomial of $(n-1)$th degree at most, which, in case no two of the a_i are equal to each other, agrees with $f(z)$ at the points $z=a_1, z=a_2, \cdots, z=a_n$, while if some of the a_i are equal to each other, it is the limit of the Lagrange interpolation polynomial corresponding to n points a_i' that are all distinct and are allowed to approach the points a_i respectively. In the latter case, if (say) a_1 occurs just n_1 times in the sequence $a_1, a_2, \cdots, a_n$, the corresponding Lagrange interpolation polynomial will have "contact" of at least order n_1-1 with $f(z)$ at $z=a_1$, that is, its derivatives of order $0, 1, \cdots, n_1-1$‡ will be equal to the corresponding derivatives of $f(z)$ at that point.

These polynomials are among the most familiar approximations to functions of a real or complex variable, and general theorems are known which prove their convergence to $f(z)$ as n becomes infinite, for properly restricted points $a_1, a_2, \cdots, a_n, \cdots$.§ The most familiar instance of these polynomials is undoubtedly the case when all a_i have a common value a, since the Lagrange polynomials corresponding to the first n terms of the sequence $a_1, a_2, \cdots, a_n, \cdots$ now reduce merely to the first n terms of the Taylor expansion of $f(z)$ about the point $z=a$; the discussion of the convergence of the polynomials to $f(z)$ belongs to the elements of the theory of functions of a complex variable. One other case where the convergence problem may be

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‡ By a derivative of order 0 will be understood, as usual, the function itself.

§ In this connection see Bieberbach's article in the Encyklopädie der mathematischen Wissenschaften, II C 4, §59.

treated with equally definite results is the case where the points a_i recur in groups of (say) m members: $a_i = a_j$ for $j \equiv i \pmod{m}$. The sequence of the resulting Lagrange polynomials turns out to be essentially equivalent to an expansion of $f(z)$ in a series of powers of $[(z-a_1)(z-a_2) \cdots (z-a_m)]$ with coefficients that are polynomials in z of degree at most $m-1$. Such expansions have been investigated by Jacobi and others.* They may be shown to converge to $f(z)$ in the largest region $|(z-a_1)(z-a_2) \cdots (z-a_n)| <$ constant, in which $f(z)$ is analytic.

2. **The polynomials $P_n(z)$.** The approximation or expansion problem to which Parts II, III of this paper are devoted is somewhat similar to the approximation problem by means of the Lagrange interpolation polynomials for the case of recurrent series $a_1, a_2, \cdots$, just mentioned.

Consider first two fixed points, a_1, a_2; $a_1 \neq a_2$. A unique polynomial, $P_{2,n}(z)$, may be shown to exist of degree $2n-1$ at most, and such that

$$(1_{2,n}) \qquad P_{2,n}^{(2i)}(a_1) = f^{(2i)}(a_1),\ P_{2,n}^{(2i)}(a_2) = f^{(2i)}(a_2) \qquad (i = 0, 1, \cdots, n-1).$$

The polynomial $P_{2,n}(z)$ resembles the Lagrange interpolation polynomial corresponding to $2n$ terms of the recurrent sequence $a_1, a_2; a_1, a_2; \cdots$ in that certain of its derivatives at a_1 and a_2 agree with the corresponding derivatives of $f(z)$; the *orders* of these derivatives, however, *differ* in the two cases. For the above Lagrange interpolation polynomial the derivatives are of order $0, 1, \cdots, n-1$; for $P_{2,n}$ they are of order $0, 2, \cdots, 2n-2$. Now in spite of this apparent similarity in definition, it will appear later that the convergence properties of $P_{2,n}(z)$ are radically different from those of the Lagrange interpolation polynomials corresponding to the sequence $a_1, a_2; a_1, a_2; \cdots$. Thus, the latter still exhibit a Taylor-like type of convergence in that they converge to $f(z)$ within the largest of a proper set of open regions (namely, the regions $|(z-a_1)(z-a_2)| <$ constant) within which $f(z)$ is analytic; the former polynomials, $P_{2,n}(z)$, however, may not converge to $f(z)$, or even may fail to converge altogether (with the possible exception of a countable set of points), even in case $f(z)$ is an integral function. Thus, for example, if $a_1 = 0$, $a_2 = \pi$, $f(z) = \sin z$, the polynomials $P_{2,n}(z)$ obviously reduce to zero identically, and consequently converge to $f(z)$ only for $z = m\pi$, where m is an integer. Again, if with the same values of a_1, a_2 we put $f(z) = \sin kz$, where k is not an integer and in absolute value greater than unity, then it turns out (see §11) that as n becomes infinite $P_{2,n}(z)$ diverges for all z except $z = m\pi$. It is thus obvious that even for integral (entire) functions, further conditions are necessary in order that

$$(2_2) \qquad \lim_{n\to\infty} P_{2,n}(z) = f(z).$$

* See Montel, *Leçons sur les Séries de Polynomes*, Paris, 1910, pp. 47, 48.

The first of the above examples, $f(z)=\sin z$, constitutes in a sense the limiting function between the set of functions for which (2_2) holds and those for which (2_2) fails to hold. More precisely, for $a_1=0$, $a_2=\pi$, a sufficient condition for the validity of (2_2), presently to be stated, is that the mode of increase of $|f(z)|$ as $|z|$ becomes infinite, should be less than that of $|\sin z|$.

Consider next m fixed points, $a_1, a_2, \cdots, a_m$, no two of which are alike, and determine a polynomial $P_{m,n}(z)$, $n=1, 2, \cdots$, of degree $mn-1$ at most, such that

$$(1_{m,n}) \qquad P_{m,n}^{(im)}(a_j) = f^{(im)}(a_j) \qquad (i = 0, 1, \cdots, n-1; j = 1, 2, \cdots, m).$$

The polynomial $P_{m,n}$ has mn available constants, while the above equations, requiring that its derivatives of order $0, m, 2m, \cdots, m(n-1)$ at the points $a_1, \cdots, a_n$ agree with the corresponding derivatives of $f(z)$, impose mn linear conditions on $P_{m,n}(z)$. These conditions may be shown to be linearly independent; hence a unique polynomial will be determined for arbitrary values of the right hand members of $(1_{m,n})$. The existence and uniqueness of $P_{m,n}$ is thus manifest.

For $m=1$ the polynomial $P_{m,n}$, as well as the Lagrange interpolation polynomial corresponding to the sequence of n terms $a_1, a_1, \cdots, a_1$, both reduce to the first n terms of the Taylor expansion of $f(z)$ about $z=a_1$. From $m=2$, the polynomials $P_{m,n}$ have just been discussed and their similarity to and difference from the Lagrange interpolation polynomials corresponding to the recurrent sequence $a_1, a_2; a_1, a_2, \cdots$ pointed out. Likewise the latter polynomials corresponding to the recurrent sequence of mn terms, $a_1, a_2, \cdots a_m; \cdots; a_1, a_2, \cdots, a_m$ resemble the polynomials $P_{m,n}$ in having n derivatives at each of the points $a_1, a_2, \cdots, a_m$ equal to the corresponding derivatives of $f(z)$. The order of the derivatives again differs in the two cases with corresponding profound differences in the nature of the convergence of $P_{m,n}(z)$ and conditions in order that

$$(2_m) \qquad \lim_{n\to\infty} P_{m,n}(z) = f(z).$$

3. **Sufficient conditions for the validity of (2_m). Necessary conditions.** Functions of a very simple type and for which $(2_m)(m>1)$ fails to hold for a general value of z, may be obtained by considering the system consisting of the differential equation

$$(3_m) \qquad d^m u(z)/dz^m - \lambda^m u(z) = 0$$

and the "boundary conditions"

$$(4_m) \qquad u(a_i) = 0 \qquad (i = 1, 2, \cdots, m).$$

The set of characteristic parameter values, that is, of values of λ for which the above system possesses a non-trivial solution, or a solution not identically zero, is readily shown to coincide with the (non-null) set of roots of a certain integral function. Any non-trivial solution of (3_m), (4_m) obviously constitutes an example for which (2_m) fails to hold for a general value of z, since $P_{m,n}(z)\equiv 0$.

Now these characteristic values are also of interest in another connection. Thus, sufficient conditions in order that (2_m) hold for all z are the following:

1. $f(z)$ is an integral function of z;
2. $f(z)$ satisfies the relation

$$(5_m) \qquad f(z) = O(e^{k|z|}), \ k < \rho_m,$$

where k is a constant, and ρ_m is the absolute value of those characteristic parameter values of (3_m), (4_m) which are nearest the origin of the λ-plane. For $m=2$ the characteristic values of (3_m), (4_m) which are nearest the origin are given by $\lambda^2=-\pi^2/(a-b)^2$, the corresponding non-trivial solutions of (3_2), (4_2) being given by $\sin[\pi(z-a)/(a-b)]$ in each case. This will be recognized as the example used in §2 for the case $a=0$, $b=\pi$. For $m=2$ the sufficiency of the conditions is shown in Theorem 1; for $m=3$ in Theorem 10, and the latter proof applies with little modification to any m.

We shall refer to an integral function $f(z)$ that satisfies a relation $f(z)=O(e^{k|z|})$ for *any* constant $k<\rho$, but for *no* constant $k>\rho$, as a function of "exponential type" ρ. The sufficient conditions just mentioned amount to requiring that $f(z)$ be of exponential type less than ρ_m.

From the consideration of a non-trivial solution of (3_m), (4_m) corresponding to a characteristic value of λ that is nearest the origin, it will be shown that in (5_m) ρ_m cannot be replaced by any larger value without incurring the failure of the conclusion for some functions $f(z)$. Consequently the above sufficient condition is the best possible one of its type. Nevertheless, this condition is not a necessary one. Thus, it is shown in Theorem 2 that, if the function $f(z)$ is odd about $(a_1+a_2)/2$, that is, if

$$f\left(\frac{a_1+a_2}{2}+z\right) = -f\left(\frac{a_1+a_2}{2}-z\right),$$

then (2_2) will hold for all z provided that $f(z)$ is merely of exponential type less than $2\rho_2$. Likewise, for $m>2$ condition (5_m) may be replaced by more lenient ones for special types of functions (§16).

For $m=2$ condition (5_m) becomes

$$f(z) = O(e^{k|z|}), \ k < \pi/|a_1 - a_2| .$$

It is of interest to point out that this condition is precisely the condition under which a theorem of F. Carlson* assures us that an integral function that vanishes at all the points congruent to a_1 (mod (a_1-a_2)) must vanish identically. A connection, though a somewhat superficial one, between the two sets of results may be seen in the fact that, as pointed out above, there exist functions for which $P_{2,n}(z)$ diverges everywhere with exception of the points congruent to a_1 (mod (a_1-a_2)); as regards convergence of $P_{2,n}(z)$, these points thus appear to play a rôle analogous to that of the origin in a series of powers of z, and are thus somewhat on a par with a_1 and a_2.†

The general question of the validity of (2_m) may obviously be separated into two parts: first, does the sequence $P_{m,n}(z)$ converge at all?; second, if it converges for a proper point set, does it converge to $f(z)$ there? To answer the first question, the nature of the convergence of $P_{m,n}(z)$ or of their equivalent series, for arbitrary values of $f^{(im)}(a_j)$, the right hand members in $(1_{m,n})$, is studied in §§9, 10, 17. It is shown that for $m=2$ the series in question either converges for all z, or diverges for all z with the possible exception of a countable set of points with ∞ as its only limit point, depending upon whether the sums

$$\sum_{n=0}^{\infty}(-1)^n[f^{(2n)}(a_1)+f^{(2n)}(a_2)][(a_1-a_2)/\pi]^{2n},$$

$$\sum_{n=0}^{\infty}(-1)^n[f^{(2n)}(a_1)-f^{(2n)}(a_2)][(a_1-a_2)/(2\pi)]^{2n}$$

both converge or not (Theorem 6). In the former case $P_{2,n}(z)$ approaches a limit uniformly in any finite region, and the limiting function is integral and may be broken up into a sum of two functions, respectively even‡ and odd

* In this connection see G. H. Hardy, *On two theorems due to F. Carlson and S. Wigert,* Acta Mathematica, vol. 42 (1920), p. 328. In Carlson's theorem the inessential restriction $a_1=0$, $a_2=1$ is made. See also P. L. Srivastava, *On a class of Taylor's series,* Annals of Mathematics, (2), vol. 30 (1928), p. 39, where the same conditions are employed.

† This suggests that an approximation problem that is more vitally connected with the problem of approximating by means of $P_{2,n}$ than the Lagrange interpolation polynomials corresponding to a sequence of recurring points (mentioned in §§1, 2) is the problem of the Stirling interpolation series. The latter, it will be recalled, is equivalent to the sequence of polynomials of degree $2n$, which agree with the function at the points $-n, -n+1, \cdots, n$. This problem has been treated among others by N. E. Nörlund in his *Leçons sur les Séries d'Interpolation,* Paris, 1926, Chapter II. Nörlund derives necessary conditions, as well as sufficient conditions, for the convergence of these interpolations to $f(z)$; these conditions are in the form of inequalities on $|f(re^{i\theta})|$ for various θ. No use has been made of such inequalities in this paper, though it is likely that an application of conditions of this type would prove fruitful.

‡ That is, satisfying $$f\left(\frac{a_1+a_2}{2}+z\right)=f\left(\frac{a_1+a_2}{2}-z\right).$$

about $(a_1+a_2)/2$, and of exponential type at most equal to ρ_2, $2\rho_2$ respectively. From these conclusions are obtained *necessary* conditions for the validity of (2_2) (Theorem 7), and these conditions turn out to be almost sufficient (Theorem 8).

Analogous results are also established for $m>2$ (Theorems 11a, 11b, 12), subject, however, to a slight restriction concerning the location of the characteristic values of (3_m), (4_m). No analogue of Theorem 8, however, has thus far been found. The somewhat greater completeness of the results for the case $m=2$ is due to the fact that many familiar notions (among them, evenness and oddness, simply periodic functions) which can be utilized for $m=2$ do not admit of immediate and obvious generalizations for higher values of m; also to the greater familiarity of the various functions encountered in the discussion. For these reasons the case $m=2$ has been dealt with separately in Part II, while the case $m=3$ is dealt with in Part III in a manner which immediately generalizes to higher values of m.

4. **The method of proof. Other expansion problems.** The procedure employed in proving the above results is not without interest in itself. The proof of the sufficient conditions is obtained by making use of certain properly defined "Green's functions" $G_{m,i}(z, s; \lambda)$, $i=1, 2, \cdots, m$, associated with the system consisting of

$$(3_m') \qquad d^m u(z)/dz^m - \lambda^m u(z) = (-1)^m v(z)$$

and of (4_m) in such a way that

$$(6_m) \qquad u(z) = \sum_{i=1}^{m} \int_z^{a_i} G_{m,i}(z, s; \lambda) v(s) ds$$

is equivalent to $(3_m')$, (4_m); here all the variables range over their complex planes, and the functions $u(z)$, $v(z)$ are analytic.* The Green's functions $G_{m,i}$ are not immediately connected with the polynomials $P_{m,n}$; however, the coefficients which result from expanding $G_{m,i}$ in powers of λ (or rather of λ^m) are very intimately connected with $P_{m,n}$. Thus it is shown that the "remainder" $f(z)-P_{m,n}(z)$ can be expressed in terms of these coefficients; this is done by means of successive applications of the formula

$$(7_m) \quad \int_{s_1}^{s_2} [u(s)v^{(m)}(s) + (-1)^{(m+1)} u^{(m)}(s) v(s)] ds = u(s)v^{(m-1)}(s) - u'(s)v^{(m-2)}(s) + \cdots + (-1)^{(m-1)} u^{(m-1)}(s) v(s) \Big]_{s_1}^{s_2};$$

* These Green's functions are also shown to be the resolvent system of an integral equation of the second kind with the same integration pattern as in (6_m).

the easily determined asymptotic behavior of the coefficients is then utilized to discuss the convergence of the remainder to zero.

For $m=1$ the above procedure is shown to lead essentially to a familiar way of establishing Taylor's series (§18). The crux of the difference between the cases $m=1$ and $m>1$ appears in the course of the proof as due to the fact that in the latter case the Green's functions are *meromorphic* in the parameter λ, while in the former case the Green's function (there is now only *one* such function) is *integral* in λ.

In discussing the necessary conditions the Green's functions $G_{i,m}$ are also utilized. Thus for $m=2$ it is shown that when $\partial G_{2,1}(z, s; \lambda)/\partial s\,|_{s=a_1}$ is expanded in powers of λ^2, the coefficient of λ^{2n} (it is denoted by $\alpha_{n-1}(z)$ in Part II) is a polynomial whose even-order derivatives at a_1, a_2 all vanish, with exception of the derivative of order $2n$ at a_1, which is equal to unity. Similar statements are true for $\beta_{n-1}(z)$, the coefficient of λ^{2n} in $\partial G_{2,2}(z, s; \lambda)/\partial s\,|_{s=a_2}$, but with a_1, a_2 interchanged. Obviously one may express $P_{2,n}(z)$ in terms of α_n, β_n thus:

$$P_{2,n}(z) = \sum_{i=0}^{n-1}[f^{2i}(a_1)\alpha_i(z) + f^{2i}(a_2)\beta_i(z)].$$

Again the asymptotic behavior of $\alpha_n(z)$, $\beta_n(z)$ is determined from the nature of the singularities of their generating functions on the circle of convergence and beyond. This aymptotic behavior is utilized in investigating the nature of the convergence of $P_{2,n}(z)$, and in deducing necessary conditions for the validity of (2_2). A similar procedure is followed for $m>2$.

The methods of proof used are highly suggestive and may be applied to a variety of approximation problems. Some of these are discussed somewhat briefly in Part IV, but no attempt is made to formulate a general theory for the present. Among the problems discussed are the following:

1. Approximations by means of solutions of certain linear differential equations with constant coefficients, where the approximating function is chosen so that its even-order derivatives of a sufficient order at two points, a_1, a_2, are equal to those of $f(z)$. These approximations are suggested by expanding the Green's functions $G_{2,i}$ connected with the polynomials $P_{2,n}(z)$ about an arbitrary value of λ, λ_0, not a pole of $G_{2,i}$. For $\lambda_0=0$ these approximations reduce to $P_{2,n}(z)$ (§19).

2. Expansions suggested by means of the Laurent expansion of the Green's functions (of a proper system with a parameter) about a pole of the parameter (§20).

3. Certain boundary value expansions of functions of n (real) variables (§22). These expansions are the n-dimensional analogues from a certain point

of view of the approximations of one real variable by means of the polynomials $P_{2,n}$.

The last one of the above expansions is similar to the boundary-value expansions considered by the author in a paper entitled *On Green's formulas for analytic functions.** In these expansions an analytic function of several real variables in a given region is expressed in terms of the boundary values of its iterated Laplacians and their normal derivatives. The present paper had its origin in the attempt to eliminate the normal derivatives and express an analytic function over a given region in terms of the boundary values of the iterated Laplacians only. The representation obtained in this way generalizes the familiar expression of a harmonic function in terms of its boundary values by means of Green's function. However, throughout most of this paper we have confined ourselves to functions of *one* variable only, but have allowed it to range over the whole complex plane.

It is suggested that for a first reading Part III be omitted, in view of its formal complexity.

PART II. THE POLYNOMIALS $P_{2,n}(z)$

5. Introduction of the polynomials $\alpha_n(z)$, $\beta_n(z)$ and of the Green's functions $A(z, s; \lambda)$, $B(z, s; \lambda)$. Throughout Part II we shall denote a_1, a_2 by a and b respectively.

It will be recalled that the polynomials $P_{2,n}(z)$ are uniquely determined by means of the conditions $(1_{2,n})$ which equate at $z=a$, $z=b$ those even-order derivatives of $P_{2,n}(z)$ that do not vanish identically, to the corresponding derivatives of $f(z)$ at the same points. Suppose now that we replace all the right hand members of $(1_{2,n})$ by zeros with the exception of $f^{(2n-2)}(a_1)$, replacing the latter by 1. The resulting polynomial $P_{2,n}$ we shall denote by $\alpha_{n-1}(z)$; from equations $(1_{2,n})$ it follows that its degree is at *most* equal to $2n-1$. Now since its $(2n-2)$th derivative takes on two different values at a and b, namely 0 and 1, respectively, it follows that α_{n-1} is at *least* of degree $2n-1$. Hence it is *actually* of degree $2n-1$. In a similar way we define polynomials $\beta_{n-1}(z)$ as the polynomial of degree $2n-1$ at most whose even-order derivatives vanish at $z=a$, $z=b$ with the exception of the $(2n-2)$th derivative at b, whose value is 1. In terms of the polynomials $\alpha_n(z)$, $\beta_n(z)$, as pointed out in §4, we may express the polynomial $P_{2,n}(z)$ as follows:

$$P_{2,n}(z) = \sum_{i=0}^{n-1}[f^{(2i)}(a)\alpha_i(z) + f^{(2i)}(b)\beta_i(z)].$$

* Presented to the Society, December, 1928.

It will sometimes be necessary to exhibit the fact that the polynomials $P_{2,n}(z)$, $\alpha_n(z)$, $\beta_n(z)$, in addition to depending upon z, also depend upon a and b. Where this is the case we shall denote them by $P_{2,n}(a, b; z)$, $\alpha_n(a, b; z)$, $\beta_n(a, b; z)$ respectively. It will often be convenient to make the restriction $a=0$, $b=\pi$. We shall write $'P_{2,n}(z)$, $'\alpha_n(z)$, $'\beta_n(z)$ in place of $P_{2,n}(0, \pi; z)$, $\alpha_n(0, \pi; z)$, $\beta_n(0, \pi; z)$ respectively; likewise we shall precede with a prime the number of any formula in which a and b have been equated to 0 and π.

We now turn to the Green's functions $G_{2,1}$, $G_{2,2}$ mentioned in §4, denoting them, however, by $A(z, s; \lambda)$, $B(z, s; \lambda)$ respectively. They are defined by means of the equations

$$\frac{\partial^2 A(z, s; \lambda)}{\partial s^2} = \lambda^2 A(z, s; \lambda), \quad \frac{\partial^2 B(z, s; \lambda)}{\partial s^2} = \lambda^2 B(z, s; \lambda); \tag{8}$$

$$A(z, a; \lambda) = 0; \tag{9$_1$}$$

$$B(z, b; \lambda) = 0; \tag{9$_2$}$$

$$A(z, z; \lambda) + B(z, z; \lambda) = 0; \tag{9$_3$}$$

$$\frac{\partial}{\partial s}[A(z, s; \lambda) + B(z, s; \lambda)]\Big|_{s=z} = 1. \tag{9$_4$}$$

From (8), (9_1), (9_2) it follows that, for $\lambda \neq 0$, A and B are of the form $\overline{A}$ sinh $\lambda(s-a)$, $\overline{B}$ sinh $\lambda(s-b)$ respectively, where $\overline{A}$, $\overline{B}$ are independent of s. Substituting in (9_3) and (9_4) we find that when the resulting linear equations are compatible their solution is given by

$$\begin{aligned} A(z, s; \lambda) &= \frac{\sinh \lambda(z-b) \sinh \lambda(s-a)}{\lambda \sinh \lambda(a-b)}, \\ B(z, s; \lambda) &= -\frac{\sinh \lambda(z-a) \sinh \lambda(s-b)}{\lambda \sinh \lambda(a-b)}. \end{aligned} \tag{10}$$

It will be observed that $\lambda = \pm n\pi i/(a-b)$, $n=1, 2, \cdots$, are (for general values of z and s) poles of A and B. Now it is precisely for these values of λ that the system (8), (9_i) is incompatible. The case $\lambda=0$ still remains to be examined. It is readily seen that, for this value of λ, (8), (9_i) possess a unique solution which is equal to the limit approached by the right hand members of (10) as λ approaches 0. With $A(z, s; 0)$, $B(z, s; 0)$ defined as equal to this limit, A and B are analytic at $\lambda=0$.

As pointed out in §4, the functions $A(z, s; \lambda)$, $B(z, s; \lambda)$ naturally arise in connection with the semi-homogeneous system

$$d^2u(z)/dz^2 - \lambda^2 u(z) = v(z), \tag{3$_2'$}$$

$$u(a) = 0,\ u(b) = 0, \tag{4$_2$}$$

where we suppose that $v(z)$ is a given analytic function, and $u(z)$ is sought analytic. It is easily verified that for values of λ which are not poles of A, B, the homogeneous system (3_2), (4_2) possesses only the solution $u(z)=0$; hence that the solution of $(3_2')$, (4_2), if it exists, is unique; and that

$$(6_2) \qquad u(z) = \int_z^a A(z, s; \lambda)v(s)ds + \int_z^b B(z, s; \lambda)v(s)ds,$$

where the integrations are carried out along any paths joining the end points and lying in the region of analyticity of $v(z)$, furnishes such a solution.

In the real domain, that is, for a and b real $(a<b)$, $v(z)$, $u(z)$ functions of the real variable z for $a\leqq z\leqq b$ of class C^0, C'' respectively, the Green's function $G(z, s; \lambda)$ of the above system is commonly defined by means of conditions "adjoint" to those of (8), (9_i), or by means of the integral representation $u(z)=\int_a^b G(z, s; \lambda)v(s)ds$. Thus, for real values of z and s, $G=-A$ for $a\leqq s\leqq z$, $G=B$ for $z\leqq s\leqq b$. Our departure from the conventions will prove convenient when it comes to the applications in the following sections.

6. **Expansions of the Green's functions in powers of λ. Expression of the remainder $f(z)-P_{2,n}(z)$ in terms of the resulting coefficients.** Of particular interest in connection with the polynomials $P_{2,n}(z)$ are the coefficients that result when A and B are expanded in powers of λ. From (10) it is obvious that A and B are even functions of λ, analytic at the origin in the λ^2-plane, and that the singularity of each that is nearest the origin is at $\lambda^2=-\pi^2/(a-b)^2$. They may therefore be expanded in powers of λ^2, the resulting expansions being valid for $|\lambda|<\pi/|a-b|$:

$$(11) \qquad A(z, s; \lambda) = \sum_{n=0}^{\infty} \lambda^{2n}A_n(z, s), \quad B(z, s; \lambda) = \sum_{n=0}^{\infty} \lambda^{2n}B_n(z, s).$$

If we now substitute these power series in (8), (9_i) and compare coefficients of like powers of λ^2 on both sides of the resulting equations, we deduce the following properties of A_n and B_n:

$$(12) \quad \frac{\partial^2 A_n(z, s)}{\partial s^2} = \begin{cases} 0 \text{ for } n = 0, \\ A_{n-1}(z, s) \text{ for } n > 0, \end{cases} \quad \frac{\partial^2 B_n(z, s)}{\partial s^2} = \begin{cases} 0 \text{ for } n = 0, \\ B_{n-1}(z, s) \text{ for } n > 0, \end{cases}$$

$$(13_1) \qquad A_n(z, a) = 0,$$

$$(13_2) \qquad B_n(z, b) = 0,$$

$$(13_3) \qquad A_n(z, z) + B_n(z, z) = 0,$$

$$(13_4) \qquad \frac{\partial}{\partial s}[A_n(z, s) + B_n(z, s)]\Big|_{s=z} = \begin{cases} 1 \text{ for } n = 0, \\ 0 \text{ for } n > 0. \end{cases}$$

To show the connection of all these Green's functions with the problem at hand we shall now apply the formula

$$(7_2)\qquad \int_{s_1}^{s_2}[u(s)v''(s)-u''(s)v(s)]ds = u(s)v'(s)-u'(s)v(s)\Big]_{s_1}^{s_2}$$

between the limits z and a to the pairs of functions $f(s)$, $A_0(z, s)$; $f''(s)$, $A_1(z, s)$; $\cdots$; $f^{(2n)}(s)$, $A_n(z, s)$, letting $f^{(2i)}(s)$ take the place of $u(s)$ and choosing a path of integration that lies inside a region in which f is analytic, and is the same for all the $n+1$ integrations. Adding the resulting equations and making use of (12) we get

$$-\int_z^a f^{(2n+2)}(s)A_n(z, s)ds = \sum_{i=0}^{n} f^{(2i)}(s)\frac{\partial A_i(z, s)}{\partial s} - f^{(2i+1)}(s)A_i(z, s)\Big]_{s=z}^{s=a}.$$

If now to this equation we add the equation

$$-\int_z^b f^{(2n+2)}(s)B_n(z, s)ds = \sum_{i=0}^{n} f^{(2i)}(s)\frac{\partial B_i(z, s)}{\partial s} - f^{(2i+1)}(s)B_i(z, s)\Big]_{s=z}^{s=b}$$

obtained in a similar way by applying (7_2) to the pairs of functions $f(s)$, $B_0(z, s)$; $\cdots$; $f^{(2n)}(s)$, $A_n(z, s)$ between the limits z and b, and simplify the right hand member by means of (13_i), we obtain, on transposing,

$$(14)\qquad \begin{aligned} f(z) = {} & \sum_{i=0}^{n}\left[f^{(2i)}(a)\frac{\partial A_i(z, s)}{\partial s}\bigg|_{s=a} + f^{(2i)}(b)\frac{\partial B_i(z, s)}{\partial s}\bigg|_{s=b}\right] \\ & + \int_z^a f^{(2n+2)}(s)A_n(z, s)ds + \int_z^b f^{(2n+2)}(s)B_n(z, s)ds. \end{aligned}$$

Putting in place of $f(z)$ in (14) the polynomials $\alpha_n(z)$, $\beta_n(z)$ we come out with

$$(15_1)\qquad \alpha_n(z) = \frac{\partial A_n(z, s)}{\partial s}\bigg|_{s=a},$$

$$(15_2)\qquad \beta_n(z) = \frac{\partial B_n(z, s)}{\partial s}\bigg|_{s=b}.$$

Hence we may write (14) in the form

$$(16)\qquad f(z) - P_{2,n+1}(z) = \int_z^a f^{(2n+2)}(s)A_n(z, s)ds + \int_z^b f^{(2n+2)}(s)B_n(z, s)ds.$$

The connection of A_n, B_n with the polynomials $\alpha_n(z)$, $\beta_n(z)$, $P_{2,n}(z)$ is now obvious.

The fundamental formula (16) shows that $A_n(z, s)$, $B_n(z, s)$ constitute the Green's functions of the system

$$d^{2n+2}u(z)/dz^{2n+2} = v(z),$$
$$u(a) = u''(a) = \cdots = u^{(2n)}(a) = u(b) = u''(b) = \cdots = u^{(2n)}(b) = 0.$$

For if we replace either member of (16) by $u(z)$ and put $v(s)$ in place of $^{(2n+2)}(s)$, equation (16) shows that

$$u(z) = \int_z^a A_n(z, s)v(s)ds + \int_z^b B_n(z, s)v(s)ds$$

furnishes the solution of the above system.

Applying this to the case $n=0$ we see that the system

$$h''(z) - g''(z) = \lambda^2 h(z),$$
$$h(a) - g(a) = 0, \quad h(b) - g(b) = 0$$

is equivalent to the integral relation

$$h(z) = g(z) + \lambda^2\left[\int_z^a A_0(z, s)h(s)ds + \int_z^b B_0(z, s)h(s)ds\right].$$

If, however, we write the differential equation of the last system in the form

$$h''(z) - g''(z) - \lambda^2[h(z) - g(z)] = \lambda^2 g(z),$$

the system becomes of the form $(3_2')$, (4_2) of the preceding section; for λ not a pole of $A(z, s; \lambda)$, $B(z, s; \lambda)$ the system is therefore equivalent to

$$h(z) = g(z) + \lambda^2\left[\int_z^a A(z, s; \lambda)g(s)ds + \int_z^b B(z, s; \lambda)g(s)ds\right].$$

Comparing the two integral relations obtained we see that $A(z, s; \lambda)$, $B(z, s; \lambda)$ constitute the resolvent system corresponding to the kernel system $A_0(z, s)$, $B_0(z, s)$. The characteristic values of the parameter of the first integral equation are precisely the poles of the resolvent system and furnish the parameter values for which the homogeneous system obtained by putting $g(z)=0$ has non-trivial solutions.

Finally, we point out the equations (17_i), whose validity for $|\lambda| < \pi/|a-b|$ follows from (10), (11), and (15_i):

$$(17_1) \qquad \left.\frac{\partial A(z, s; \lambda)}{\partial s}\right|_{s=a} = \frac{\sinh \lambda(z-b)}{\sinh \lambda(a-b)} = \sum_{n=0}^{\infty} \lambda^{2n}\alpha_n(z),$$

$$(17_2) \qquad \left.\frac{\partial B(z, s; \lambda)}{\partial s}\right|_{s=b} = \frac{\sinh \lambda(z-a)}{\sinh \lambda(a-b)} = \sum_{n=0}^{\infty} \lambda^{2n}\beta_n(z).*$$

* From this it is seen that $d^2\alpha_n(z)/dz^2$ is equal to $\alpha_{n-1}(z)$ for $n>0$, and to 0 for $n=0$; similar relations hold for $\beta_n(z)$.

7. **Asymptotic formulas for A_n, B_n, α_n, β_n.** In this section we shall establish certain asymptotic properties of the functions listed in the title, for large n; these properties will be utilized further on. The method used is based on examining the singularities of the generating functions (11), (17_i). It is essentially the method employed by Darboux in his article *Mêmoire sur l'approximation des fonctions de très-grands nombres* etc.* We shall confine ourselves to the special case $a=0$, $b=\pi$. The Green's functions A, B, A_n, B_n, corresponding to these values of a and b, we shall indicate by $'A$, $'B$, $'A_n$, $'B_n$, in accordance with the notation explained in §5.

The functions $'A$, $'B$ possess simple poles at $\lambda=\pm mi$, $m=1, 2, \cdots$. Direct computation shows that the sum of the principal parts of $'A$ at the two poles $\lambda=mi$, $\lambda=-mi$ is equal to

$$(18_1)\qquad \frac{2}{\pi}(\sin mz)(\sin ms)\frac{1}{\lambda^2+m^2}\cdot$$

Likewise

$$(18_2)\qquad -\frac{2}{\pi}(\sin mz)(\sin ms)\frac{1}{\lambda^2+m^2}$$

is equal to the sum of the principal parts of $'B$ at $\lambda=mi$, $\lambda=-mi$.† Hence

$$(19_1)\qquad 'A(z, s; \lambda) - \frac{2}{\pi}\sum_{m=1}^{M-1}\frac{\sin mz \sin ms}{\lambda^2+m^2},$$

$$(19_2)\qquad 'B(z, s; \lambda) + \frac{2}{\pi}\sum_{m=1}^{M-1}\frac{\sin mz \sin ms}{\lambda^2+m^2}$$

are analytic throughout $|\lambda|<M-\epsilon$, $\epsilon>0$. Using this fact we see that

$$(20)\qquad 'A_n(z, s), -'B_n(z, s) = (-1)^n\frac{2}{\pi}\sum_{m=1}^{M-1}\frac{\sin mz \sin ms}{m^{2(n+1)}} + O[(M-\epsilon)^{-2n}], \epsilon>0,$$

where the order relations for a fixed M hold uniformly for z and s ranging over any finite part of their planes.

To obtain similar formulas for $'\alpha_n(z)$, $'\beta_n(z)$ we compute the principal parts of the generating functions in (17_i) by differentiating (19_i). We find

$$(21_1)\qquad '\alpha_n(z) = (-1)^n\frac{2}{\pi}\sum_{m=1}^{M-1}\frac{\sin mz}{m^{2n+1}} + O[(M-\epsilon)^{-2n}],$$

$$(21_2)\qquad '\beta_n(z) = (-1)^n\frac{2}{\pi}\sum_{m=1}^{M-1}(-1)^{m+1}\frac{\sin mz}{m^{2n+1}} + O[(M-\epsilon)^{-2n}], \epsilon>0,$$

* Journal de Mathématiques, (3), vol. 4 (1878), p. 1, p. 377.

† The latter computation might be avoided by noting that $A(z, s; \lambda)+B(z, s; \lambda)$ is integral in λ, a fact inferred from (8), (9_3), (9_4) or directly from (10).

where for a fixed M the order relations hold uniformly in z ranging over any finite part of its plane.

For a fixed n the above order relations have not been shown to hold uniformly in M; hence no information is gained by letting M become infinite while n is held fixed. The behavior of the resulting Fourier sine series is not, however, without interest. For general values of z and s they fail to converge. However, for real values of z and s convergence does take place, and we have for $0 \leqq s, z \leqq \pi$

$$\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin mz \sin ms}{\lambda^2 + m^2} = \begin{cases} -\,'A(z, s; \lambda) & \text{for } s \leqq z, \\ 'B(z, s; \lambda) & \text{for } s \geqq z, \end{cases}$$

$$(-1)^n \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin mz \sin ms}{m^{2n+2}} = \begin{cases} -\,'A_n(z, s) & \text{for } s \leqq z, \\ 'B_n(z, s) & \text{for } s \geqq z.* \end{cases}$$

For a general position of a and b analogous results may be obtained by effecting in the s- and z-planes a linear integral transformation that sends 0 and π into a and b respectively, and at the same time multiplying λ by $(b-a)/\pi$.

8. **Sufficient conditions for the convergence of $P_{2,n}(z)$ to $f(z)$.** We shall now prove

THEOREM 1. *If $f(z)$ is an integral function satisfying*

$$f(z) = O(e^{k|z|}), \quad k < \pi/\,|\,a - b\,|\,, \tag{22}$$

then $P_{2,n}(z)$ converges to $f(z)$ for all z, the convergence being uniform in any finite region.

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* These facts may be established as follows. Let $0 \leqq s, z \leqq \pi$ and let

$$K(z, s) = \begin{cases} -\,'A_0(z, s) & \text{for } s \leqq z, \\ 'B_0(z, s) & \text{for } s \geqq z, \end{cases}$$

and

$$G(z, s; \lambda) = \begin{cases} -\,'A(z, s; \lambda) \text{ for } s \leqq z, \\ 'B(z, s; \lambda) \text{ for } s \geqq z. \end{cases}$$

Then, from what has been explained toward the end of §6, $G(z, s; \lambda)$ is seen to be the resolvent to the symmetric kernel $K(z, s)$ in the ordinary sense. The characteristic values of the parameter λ^2 are the values $\lambda^2 = -m^2\pi^2$ with $(2/\pi)^{1/2} \sin mz$ as the only linearly independent corresponding characteristic function in normalised form. Making use of a familiar bilinear form for the resolvent of a symmetric kernel (see, for instance, A. Kneser, *Die Integralgleichungen*, Braunschweig, 1922, Chapter III) we see that the first sine series in the text is the Fourier sine series of $G(z, s; \lambda)$, and hence infer the validity of the first equation. The second equation now follows by equating coefficients of like powers of λ^2 on both sides.

THEOREM 2. *If $f(z)$ is an integral function which is odd* about $(a+b)/2$, and satisfies*

$$(23)\qquad f(z) = O(e^{k|z|}), \quad k < 2\pi/|a-b|,$$

then $P_{2,n}(z)$ converges to $f(z)$ as in Theorem 1.

Combining these theorems we obtain

THEOREM 3. *Let $f(z)$ be an integral function; resolve it into a sum of two functions, $e(z)+o(z)$, where $e(z)$ is even about $(a+b)/2$ and $o(z)$ is odd about $(a+b)/2$, and suppose that $e(z)$ satisfies* (22) *while $o(z)$ satisfies* (23); *then $P_{2,n}(z)$ converges to $f(z)$ as in Theorem* 1.

Using a term explained in §3, the exponential conditions of, say, Theorem 3 are that $e(z)$, $o(z)$ are of "exponential type" less than $\pi/|a-b|$, $2\pi/|a-b|$ respectively.

It may be remarked at the outset that the constants of the right hand inequalities in (22) and (23) may not be replaced by any larger values. This may be seen for Theorem 1 by considering the example $a=0$, $b=\pi$, $f(z)=\sin z$, already mentioned in §2, and for Theorem 2 by taking the same values of a and b and putting $f(z)=\sin 2z$.

It will be proved in §11 (Theorem 7) that a *necessary* condition for the convergence of $P_{2,n}(z)$ to $f(z)$ is that $e(z)$, $o(z)$ be of exponential types less than or equal to ρ_2 and $2\rho_2$ respectively.

The constant $\pi/|a-b|$ is precisely the same as the constant ρ_2 of the inequality (5_2), while the function $\sin z$ constitutes in fact the non-trivial solution of the system (3_2), (4_2) corresponding to the characteristic parameter value which is nearest the origin.

To prove Theorem 1 we shall first suppose that $a=0$, $b=\pi$. This will simplify the formal work. The general case may be reduced to this special case by means of the linear integral transformation of z mentioned at the end of the preceding section. As a result of this transformation (22) becomes equivalent to

$$('22)\qquad f(z) = O(e^{k|z|}), \quad k < 1.$$

We shall prove that under this condition the remainder $f(z) - {}'P_{2,n}(z)$ approaches zero uniformly.

* As explained in §3, a function $f(z)$ will be said to be "odd about a point $z=c$" if it satisfies $f(c+z) = -f(c-z)$ for an arbitrary z, that is, if all its even-order derivatives at c vanish. Similarly, if $f(z)$ satisfies $f(c+z)=f(c-z)$, that is, if all its odd-order derivatives vanish at c, $f(z)$ will be said to be "even about $z=c$."

Let $f(z)=\sum_{n=0}^{\infty}c_n z^n$. Applying Cauchy's integral formula over the circle $|z|=r$* and using ('22), we get

$$|c_n| < Ce^{kr}/r^n,$$

where $k<1$ and C is a constant†; and replacing e^{kr}/r^n by its minimum value, obtain

$$|c_n| < C(ke/n)^n.$$

Hence

$$|f^{(n)}(0)| < Cn!(ke/n)^n,$$

and, introducing Stirling's formula,

$$|f^{(n)}(0)| < Ce^{-n}n^{n+(1/2)}(2\pi)^{1/2}[1 + O(1/n)](ke/n)^n = Cn^{1/2}k^n[1 + O(1/n)];$$

hence

$$|f^{(n)}(0)| < Cn^{1/2}k^n.$$

A similar inequality

(24) $$|f^{(n)}(z)| < Cn^{1/2}k^n$$

holds for z in an arbitrary finite region, and with the constant C dependent only on the region but independent of the position of z in it. For from ('22) follows

$$|f(z + z')| < Ce^{k|z|}e^{k|z'|}.$$

Now the product of the first two factors is bounded if z lies in a prescribed finite region. Applying Cauchy's integral over a circle with center at z and radius z', and proceeding as above, one arrives at (24).

We now turn to $'A_n(z, s)$, $'B_n(z, s)$ and make use of (20) with M equated to 2. We obtain

$$'A_n(z, s), \ -\ 'B_n(z, s) = (-1)^n(2/\pi)\sin z \sin s + O[(2-\epsilon)^{-2n}],\ \epsilon > 0.$$

Utilizing this result, as well as (24) with n replaced by $2n+2$, in the integral representation (16) for the remainder $f(z)-P_{2,n+1}(z)$, we see that for z and s ranging over any finite regions of their respective planes the integrands in (16) may be made in absolute value less than a prescribed constant by choosing n large enough. Hence the remainder approaches zero uniformly in z as n becomes infinite; the proof of Theorem 1 is thus complete.

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* This part of the proof is adapted from Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, 1927, p. 228.

† Wherever that will lead to no confusion the same letter C will be used to denote different constants in different inequalities.

Several remarks are of interest at this stage. In the first place it is obvious that the exponential function of $|z|$ employed may be replaced by any function of $|z|$, $F(|z|)$, such that

$$\min\,[F(r)/r^{2n}] = o[1/(2n)!],$$

where the left hand member represents the minimum of the bracket for all positive r for a fixed integer value of n.

We next recall again that the conditions of Theorem 1 are the same as those of a theorem of Carlson regarding functions which vanish at all the points congruent to a (mod $(a-b)$) (see §3). A slight connection between the two sets of results may now be seen from the asymptotic representation (21_i). Putting $M=2$ we get

$$'\alpha_n(z),\ '\beta_n(z) = (-1)^n(2/\pi)\sin z + O[(2-\epsilon)^{-2n}],\ \epsilon > 0.$$

Thus the roots of $\alpha_n(z)$, $\beta_n(z)$ approach asymptotically the points congruent to a (mod $(a-b)$). This shows that the latter points are in a sense equivalent to a and b, and makes plausible the existence of functions for which $P_{2,n}(z)$ converges to $f(z)$ for these points while diverging for any other value of z. An example of a function of this kind is given in §11.

We now turn to Theorem 2. It will be noticed that the constant figuring in the right hand member of the second inequality in (23) is twice as large as the corresponding constant in (22).* Nevertheless, Theorem 2 may be deduced from Theorem 1 as follows.

Instead of forming the polynomials $P_{2,n}(z)$ corresponding to a, b or $P_{2,n}(a,\ b;\ z)$ suppose we form $P_{2,n}(a,\ (a+b)/2;\ z)$. At $z=(a+b)/2$, all the even-order derivatives of $f(z)$ vanish. Hence all the even-order derivatives of $P_{2,n}(a,\ (a+b)/2;\ z)$ will vanish at $z=(a+b)/2$; this polynomial will consequently be odd about $(a+b)/2$ and its derivatives at $z=b$ of orders $0, 2, \cdots, 2n-2$ will be equal to the corresponding derivatives of $f(z)$ at the same point. Therefore for the odd functions in question

$$P_{2,n}(a,\ (a+b)/2;\ z) = P_{2,n}(a,\ b;\ z).$$

Applying Theorem 1 to the polynomials on the left we get the result desired.

For the sake of the analogues of Theorem 2 in Part III (Theorems 11a, 11b) we point out that the conditions of this theorem are equivalent to

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* This, of course, has been rendered possible only by restricting the range of functions to functions odd about $(a+b)/2$.

1. $$f(z) = O(e^{k|z|}),\ k < \frac{2\pi}{|a-b|},$$

2. $$f^{2n}(a) + f^{2n}(b) = 0 \qquad (n = 0, 1, 2, \cdots),$$

3. $$\int_a^b f(s) \sin \frac{\pi(s-a)}{a-b}\, ds = 0.$$

This may be shown by breaking up $f(z)$ into a sum of two functions, one of which is even about $(a+b)/2$ and the other odd, and proving that as a consequence of the above conditions the even component must reduce to zero. Since the odd component satisfies condition 2 automatically, the even one must. Hence its even-order derivatives at a and b vanish. Consequently it is periodic of period $2(a-b)$. As a consequence of condition 1 and its periodicity, it may be shown to reduce to

$$C \sin \frac{\pi(z-a)}{a-b}$$

(see proof of Theorem 8 in §11). Finally, condition 3 reduces C to zero.

It is of interest to point out that Carlson's theorem possesses no extension forming a counterpart of Theorem 2 for functions which are odd about $(a+b)/2$, since there exist functions satisfying the conditions of Theorem 2 which vanish at all the points congruent to a (mod $(a-b)$) without vanishing identically. An example of this kind is given by $a=0$, $b=\pi$, $f(z)=[(\pi/2)-z]$ $\sin z$.

9. Convergence of the series $\sum[c_n\alpha_n(z)+d_n\beta_n(z)]$ for special cases. The theorems of the preceding section give sufficient conditions for the convergence of $P_{2,n}(z)$ to $f(z)$ or for the validity of

$$f(z) = \sum_{n=0}^{\infty}[f^{(2n)}(a)\alpha_n(z) + f^{(2n)}(b)\beta_n(z)].$$

As pointed out in §3, the question of the validity of this equation may conveniently be broken up into two parts: first the question of convergence merely; second, the question of equality of the limit to $f(z)$. To answer these questions we shall consider a series $\sum_{n=0}^{\infty}[c_n\alpha_n(z)+d_n\beta_n(z)]$, where c_n, d_n are arbitrary constants, and examine the convergence of such a series and the nature of the sum function. In this section we shall only consider certain special cases covered by Theorems 4 and 5, reserving the general case for the following section.

THEOREM 4. *The series*

$$(25_1)\qquad \sum_{n=0}^{\infty} c_n\alpha_n(z),$$

$$(25_2)\qquad \sum_{n=0}^{\infty} c_n\beta_n(z),$$

$$(25_3)\qquad \sum_{n=0}^{\infty} c_n[\alpha_n(z) + \beta_n(z)]$$

either converge for all z, or else diverge for all z with the possible exception of some or all of the points congruent to a (mod $(a-b)$) *depending upon whether the series*

$$(26)\qquad \sum_{n=0}^{\infty}(-1)^n c_n[(a-b)/\pi]^{2n}$$

converges or not. In the former case the series (25_i) *converge uniformly over any finite region of the z-plane to integral functions* $c_1(z)$, $c_2(z)$, $c_3(z)$ *respectively, of exponential type at most equal to* $\pi/|a-b|$, *that is, to functions* $c_i(z)$ *satisfying*

$$(27)\qquad c_i(z) = O(e^{k|z|}) \text{ for any } k > \pi/|a-b|.$$

THEOREM 5. *The series*

$$(28)\qquad \sum_{n=0}^{\infty} c_n[\alpha_n(z) - \beta_n(z)]$$

either converges for all z, or diverges for all z with the possible exception of all or some of the points congruent to a (mod $(a-b)/2$) *depending upon whether the ies*

$$(29)\qquad \sum_{n=0}^{\infty}(-1)^n c_n[(a-b)/(2\pi)]^{2n}$$

converges or not. In the former case (28) *converges uniformly in any finite region to an integral function* $c(z)$, *odd about* $(a+b)/2$, *of exponential type at most equal to* $2\pi/|a-b|$, *that is, to a function* $c(z)$ *satisfying*

$$(30)\qquad c(z) = O(e^{k|z|}) \text{ for any } k > 2\pi/|a-b|.$$

To prove Theorem 4 consider first the series (25_1) for the special case $a=0$, $b=\pi$:

$$('25_1)\qquad \sum c_n'\alpha_n(z),$$

and suppose that it converges for a value z_0 of z, different from $n\pi$, $n = \cdots, -1, 0, 1, 2, \cdots$. Recall the asymptotic representation

$$'\alpha_n(z) = (-1)^n(2/\pi)\sin z + O[(2-\epsilon)^{-2n}],\ \epsilon > 0,$$

employed in the preceding section. From it we conclude that $'\alpha_n(z_0)(-1)^n$ approaches $(2/\pi)\sin z_0 \neq 0$ as n becomes infinite. Hence and because the series $\sum c_n\alpha_n(z_0)$ converges, it follows that the coefficients c_n are bounded. If, therefore, we break up ('25_1) into $\sum(-1)^n(z/\pi)c_n \sin z + \sum c_n O[(2-\epsilon)^{-2n}]$, choosing $\epsilon < 1$, the latter sum will converge absolutely and uniformly in any finite region of the z-plane. Consequently, from the convergence of ('25_1) for $z = z_0$ follows the convergence of $\sum(-1)^n c_n$.

Conversely, if $\sum(-1)^n c_n$ converges, the constants c_n are bounded, and each of the two sums into which ('25_1) has been broken up is seen to converge uniformly. Since the convergence is uniform over any finite region, the limit function $'c_1(z)$ is an integral function of z.

To complete the proof of Theorem 4 for the series ('25_1) it remains to show that

$$('27)\qquad 'c_1(z) = O(e^{k|z|}),$$

where k is *any* constant greater than 1. To that end we shall express the sum of the first $N+1$ terms of ('25_1) as a contour integral in the λ^2-plane.

From (17_1) it follows that

$$'\alpha_n(z) = -\frac{1}{2\pi i}\int_\gamma \frac{\sinh\lambda(z-\pi)}{\sinh\lambda\pi}\,\frac{d(\lambda^2)}{(\lambda^2)^{n+1}},$$

where the integration is carried out in the λ^2-plane over a closed path γ that goes once around the origin in a positive sense, but fails to enclose the points $\lambda^2 = -1, -4, \cdots, -n^2, \cdots$. We may therefore write

$$\sum_{n=0}^{N} c_n{}'\alpha_n(z) = \frac{1}{2\pi i}\int_\gamma -\frac{\sinh\lambda(z-\pi)}{\sinh\lambda\pi}\left(\frac{c_0}{\lambda^2}+\frac{c_1}{\lambda^4}+\cdots+\frac{c_N}{\lambda^{2N+2}}\right)d(\lambda^2).$$

Suppose, however, that we replace the path of integration by the circle $|\lambda^2| = k^2$, $1 < k < 2$; the values of both members of the last equation will then alter by an amount equal to the residue of the integrand at the pole $\lambda^2 = -1$. Since $-\sinh\lambda(z-\pi)/\sinh\lambda\pi - (2/\pi)(\sin z)/(1+\lambda^2)$ is analytic for $|\lambda^2| < 4$ (see §7), this residue has the value $(2/\pi)\sin z[c_0 - c_1 + \cdots + (-1)^N c_N]$. Therefore

$$\sum_{n=0}^{N} c_n\alpha_n(z) = \frac{1}{2\pi i}\int_{|\lambda^2|=k^2} -\frac{\sinh\lambda(z-\pi)}{\sinh\lambda\pi}\left(\frac{c_0}{\lambda^2}+\frac{c_1}{\lambda^4}+\cdots+\frac{c_N}{\lambda^{2N+2}}\right)d(\lambda^2)$$
$$+\frac{2}{\pi}\sin z[c_0 - c_1 + \cdots + (-1)^N c_N].$$

Now since we are dealing with the case where $\sum(-1)^n c_n$ converges, the bracket above will converge as N becomes infinite, to a proper limit γ_0. Consequently the power series in $1/\lambda^2$, $c_0/\lambda^2+c_1/\lambda^4+c_2/\lambda^6+\cdots$ converges at least for $|\lambda^2|>1$ to a certain analytic function of $1/\lambda^2$, $\phi(1/\lambda^2)$. If therefore we let N become infinite, the integrand on the right will converge uniformly over the circle of integration, and we shall get, in the limit,

$$'c_1(z) = \frac{1}{2\pi i}\int -\frac{\sinh\lambda(z-\pi)}{\sinh\lambda\pi}\phi(1/\lambda^2)d(\lambda^2) + \frac{2\gamma_0}{\pi}\sin z.$$

Denoting by M the maximum of $|\phi(1/\lambda^2)/\sinh\lambda\pi|$ for $|\lambda|=k$ we have along the circle of integration $|\lambda^2|=k^2$,

$$\begin{aligned}|\sinh\lambda(z-\pi)\phi(1/\lambda^2)/\sinh\lambda\pi| &\leqq M|\sinh\lambda(z-\pi)|\\ &\leqq M\sinh|\lambda(z-\pi)| = M\sinh(k|z-\pi|),\end{aligned}$$

and hence

$$|'c_1(z)| \leqq k^2 M\sinh(k|z-\pi|) + 2|\gamma_0\sin z|/\pi.$$

From this the existence of a constant C for which $|'c_1(z)|<Ce^{k|z|}$ holds is obvious. ('27) has thus been proved for values of k between 1 and 2. Hence the proof of Theorem 4 for the series ('25_1) is now complete.

Still confining ourselves to the case $a=0$, $b=\pi$ we may prove Theorem 4 for the series (25_2), (25_3) in a similar fashion. The general case of arbitrary a and b (for all three series (25_i)) may be reduced to the case $a=0$, $b=\pi$ by means of a proper integral linear transformation of z.

To establish Theorem 5 we may employ the asymptotic representation

$$'\alpha_n(z) - '\beta_n(z) = (-1)^n(4/\pi)\sin(2z/2^{2n+1}) + O[(4-\epsilon)^{-2n}],\ \epsilon > 0,$$

obtained from (21_i) by putting $M=4$, and proceed as above, choosing, however, for the path of integration a circle $|\lambda^2|=k^2$, $2<k<3$. An easier procedure, however, is to make use of the relation

$$\alpha_n(a, b; z) - \beta_n(a, b; z) = \alpha_n(a, (a+b)/2; z)$$

by means of which series (28) is converted into a series of type (25_1); Theorem 5 now follows by an application of Theorem 4. The truth of the above relation may be rendered obvious by evaluating the even-order derivatives of the left hand member at $z=a$, $z=(a+b)/2$, at the latter point making use of (17_i).

As a consequence of the uniform convergence of the series (25_i), (28), it follows that they may be differentiated term by term, and hence that the even-order derivatives of the sum functions at $z=a$, $z=b$ are equal to a proper coefficient c_n or to 0.

10. Convergence of the series $\sum[c_n\alpha_n(z)+d_n\beta_n(z)]$ in the general case. We shall now consider the series in the title for arbitrary coefficients and prove

THEOREM 6. *Consider the series*

$$\sum_{n=0}^{\infty}\left[c_n\alpha_n(z) + d_n\beta_n(z)\right]. \tag{31}$$

If the two series

$$\sum_{n=0}^{\infty}(-1)^n(c_n + d_n)\left[(a-b)/\pi\right]^{2n}, \tag{32_1}$$

$$\sum_{n=0}^{\infty}(-1)^n(c_n - d_n)\left[(a-b)/(2\pi)\right]^{2n} \tag{32_2}$$

both converge, then the series (31) *converges uniformly in any finite region, and may be broken up into the sum of*

$$\sum_{n=0}^{\infty}(c_n + d_n)\left[\alpha_n(z) + \beta_n(z)\right]/2, \tag{33_1}$$

$$\sum_{n=0}^{\infty}(c_n - d_n)\left[\alpha_n(z) - \beta_n(z)\right]/2, \tag{33_2}$$

both series converging uniformly to sum functions respectively even and odd about $(a+b)/2$ *and in turn satisfying the relations* (27) *and* (30) *of Theorems* 4 *and* 5, *that is, of exponential types at most equal to* $\pi/|a-b|$, $2\pi/|a-b|$.

If the two series (32_i) *are not both convergent, then* (31) *diverges for all* z *with the possible exception of some or all of the roots of*

$$\sin\left[\pi(z-a)/(a-b)\right]\left\{\cos\left[\pi(z-a)/(a-b)\right] - \cos\left[\pi(z_1-a)/(a-b)\right]\right\} = 0, \tag{34}$$

where z_1 *is a fixed point.*

As in the preceding section, we may confine ourselves to the special case $a=0$, $b=\pi$, since the general case may be reduced to it by means of a proper linear integral transformation of z. Suppose that the series

$$\sum_{n=0}^{\infty}c_n'\alpha_n(z) + d_n'\beta_n(z) \tag{'31}$$

converges for $z=z_1$, $z=z_2$, where z_1, z_2 do *not* satisfy the equation

$$\begin{vmatrix} \sin z_1 & \sin z_2 \\ \sin 2z_1 & \sin 2z_2 \end{vmatrix} = 2\sin z_1 \sin z_2(\cos z_2 - \cos z_1) = 0. \tag{'34}$$

We shall write ('31) in the form

$$('31')\quad \sum_{n=0}^{\infty}\{(c_n+d_n)['\alpha_n(z)+'\beta_n(z)]+(c_n-d_n)['\alpha_n(z)-'\beta_n(z)]\}/2.$$

As this series converges for $z=z_1$, $z=z_2$, the nth term must approach zero for these values of z:

$$(c_n+d_n)['\alpha_n(z_1)+'\beta_n(z_1)]+(c_n-d_n)['\alpha_n(z_1)-'\beta_n(z_1)]=\epsilon_{1,n},$$
$$(c_n+d_n)['\alpha_n(z_2)+'\beta_n(z_2)]+(c_n-d_n)['\alpha_n(z_2)-'\beta_n(z_2)]=\epsilon_{2,n},$$

where both $\epsilon_{1,n}$ and $\epsilon_{2,n}$ approach zero as n becomes infinite. We shall consider the above as two linear equations for

$$(-1)^n(c_n+d_n),\ (-1)^n(c_n-d_n)/2^{2n}.$$

Upon recalling the formulas (21_1), (21_2) it is seen that the coefficients of these quantities approach the proper terms of the matrix

$$\left\|\begin{matrix}(4/\pi)\sin z_1 & (2/\pi)\sin 2z_1\\ (4/\pi)\sin z_2 & (2/\pi)\sin 2z_2\end{matrix}\right\|$$

as n becomes infinite. Since the determinant of this matrix does not vanish, we may, for sufficiently large n, solve the above linear equations for $(-1)^n\cdot(c_n+d_n)$, $(-1)^n(c_n-d_n)/2^{2n}$, and conclude that

$$\lim_{n\to\infty}(c_n+d_n)=\lim_{n\to\infty}(c_n-d_n)/2^{2n}=0.$$

Return now to the series ('31'), and, utilizing the formulas (21_1), (21_2), write its general term in the form

$$\frac{(-1)^n}{\pi}(c_n+d_n)\{\sin z+O[(3-\epsilon)^{-2n}]\}+\frac{(-1)^n}{\pi}\frac{(c_n-d_n)}{2^{2n+1}}\{\sin 2z+O[(2-\epsilon)^{-2n}]\}.$$

Since c_n+d_n, $(c_n-d_n)/2^{2n}$ are both bounded, it follows that the series made up of these parts of the above terms which involves the O's, converges for all finite z. Hence for any z for which ('31') converges, the series

$$\sum_{n=0}^{\infty}\left[(-1)^n(c_n+d_n)\sin z+(-1)^n\frac{c_n-d_n}{2^{2n+1}}\sin 2z\right]$$

will also converge; therefore

$$\sum_{n=0}^{\infty}\left[(-1)^n(c_n+d_n)\sin z_1+(-1)^n\frac{c_n-d_n}{2^{2n+1}}\sin 2z_1\right],$$

$$\sum_{n=0}^{\infty}\left[(-1)^n(c_n+d_n)\sin z_2+(-1)^n\frac{c_n-d_n}{2^{2n+1}}\sin 2z_2\right]$$

are both convergent. Now multiply these equations by x_1, x_2 respectively, where x_1, x_2 are the solutions of

$$2x_1 \sin z_1 + x_2 \sin 2z_1 = 1, \qquad 2x_1 \sin z_2 + x_2 \sin 2z_2 = 0,$$

and we arrive at the result that $\sum(-1)^n (c_n+d_n)$ is convergent. Likewise, by interchanging 0 and 1 above, one proves that $\sum(-1)^n (c_n-d_n)/2^{2n}$ is convergent.

Conversely, if both of these series are convergent, then by applying Theorems 4 and 5, one proves that the series (33_i) converge for all z, and to functions specified in the statement of Theorem 6.

The restriction on the two points z_1, z_2 for which ('31) cannot converge without converging everywhere is that ('34) should not be satisfied; that is, that neither should be at $n\pi$, and that $\cos z_1 \neq \cos z_2$. One of them, say z_1, may therefore be chosen at random and z_2 may be taken anywhere except for the roots of $\cos z - \cos z_1 = 0$. These restrictions are thus equivalent to the restrictions of the last sentence of Theorem 6.

11. **Necessary conditions for the convergence of $P_{2,n}(z)$ to $f(z)$. Examples. Certain existence theorems.** The conditions of Theorem 6 may be used as *necessary* conditions in order that $P_{2,n}(z)$ converge to $f(z)$ for all z, by applying them to the case where in (31) we put $c_n = f^{(2n)}(a)$, $d_n = f^{(2n)}(b)$. The result might be formulated as a special theorem as follows:

THEOREM 7. *Let $f(z) = e(z) + o(z)$, where $e(z)$ is even, and $o(z)$ odd about $(a+b)/2$. In order that*

$$(2_2) \qquad \lim_{n\to\infty} P_{2,n}(z) = \sum_{n=0}^{\infty}[f^{(2n)}(a)\alpha_n(z) + f^{(2n)}(b)\beta_n(z)] = f(z)$$

hold for all z, it is necessary that

(1) *$f(z)$ be integral;*

(2) *$e(z)$, $o(z)$ be of exponential types at most equal to $\pi/|a-b|$, $2\pi/|a-b|$;*

(3) *the two series*

$$\sum_{n=0}^{\infty}(-1)^n[f^{(2n)}(a) + f^{(2n)}(b)][(a-b)/\pi]^{2n},$$

$$\sum_{n=0}^{\infty}(-1)^n[f^{(2n)}(a) - f^{(2n)}(b)][(a-b)/(2\pi)]^{2n}$$

*both converge.**

* At this stage we take the opportunity of correcting an error that has crept into a previous publication of some of the above results in the Proceedings of the National Academy of Sciences, vol. 16 (1930), No. 1, p. 84, where the brackets $[(a-b)/\pi]$, $[(a-b)/(2\pi)]$ were erroneously replaced by their reciprocals.

The converse of Theorem 7, however, does not hold; that is, if $f(z)$ satisfies the three conditions of Theorem 7, then (2_2) need not be true. For suppose that $f(z)$ satisfies conditions 1, 2, 3, of Theorem 7; the function $f(z)+D\sin[(z-a)/(a-b)]+D'\sin[2(z-a)/(a-b)]$, where D, D' are arbitrary constants, will also satisfy the same conditions and will have the same even-order derivatives at a and b as $f(z)$. Thus a two-parameter family of functions will possess the same polynomials $P_{2,n}(z)$ and (2_2) could hold only for one member of the family at most. Theorem 8 will show, however, that this is the *only* extent to which conditions 1, 2, 3 of Theorem 7 fail to be sufficient to insure the validity of (2_2).

THEOREM 8. *If $f(z)$ satisfies conditions* 1, 2, 3, *of Theorem* 7, *then $P_{2,n}(z)$ will converge to $f(z)+D\sin[(z-a)/(a-b)]+D'\sin[2(z-a)/(a-b)]$ uniformly for z in any finite region, where D, D' are proper constants.*

That $P_{2,n}(z)$ does converge uniformly to a proper limiting function $c(z)$ follows from the convergence of the two sums in the third condition of Theorem 7 by applying Theorem 6. It also follows from the latter theorem that we may break up $\sum[f^{(2n)}(a)\alpha_n(z)+f^{(2n)}(b)\beta_n(z)]$ into a sum of two series after the manner of (33_1), (33_2), these series converging respectively to $c_e(z)$, $c_o(z)$, respectively even and odd about $(a+b)/2$, and satisfying in turn (27) and (30). Now the even-order derivatives of $c_e(z)$ at $z=a$, $z=b$ are the same as the corresponding derivatives of $e(z)$. The difference of these two functions is thus odd about both a and b and consequently periodic of period $2(a-b)$; it also satisfies the inequality (27). If then we apply the conformal transformation $z'=e^{\pi iz/(a-b)}$, $e(z)-c_e(z)$ becomes a single-valued function of z' admitting no singularities in the z'-plane except possibly for poles of the first order at $z'=0$ and at $z'=\infty$. Hence

$$e(z) - c_e(z) = C + C'e^{\pi i(z-a)/(a-b)} + C''e^{-\pi i(z-a)/(a-b)},$$

where C, C', C'' are proper constants. Equating $e(z)-c_e(z)$ to 0 for $z=a$, $z=b$, we find that $C=0$, $C'=-C''$. Hence

$$e(z) - c_e(z) = D\sin[(z-a)/(a-b)].$$

In a similar way one proves

$$o(z) - c_o(z) = D'\sin[2(z-a)/(a-b)].$$

Adding the last two equations we obtain the conclusion of Theorem 8.

It will be observed that the gap between the exponential type conditions of Theorem 3 and those of Theorem 7 or 8 is filled by functions $f(z)$ for which one or both of the following hold true:

1. The exponential type of $e(z)$ is $\pi/|a-b|$.*

2. The exponential type of $o(z)$ is $2\pi/|a-b|$. What can be said of the convergence of $P_{2,n}(z)$ for functions lying in this gap? We shall show by explicit examples that for such functions $P_{2,n}(z)$ may converge to $f(z)$, or, again, may essentially diverge. Hence the third condition of Theorem 7 is independent of the second condition.

Choose $a=0$, $b=\pi$, and put $f(z)=\sin kz$, where $|k|=1$, $k\neq 1, -1$. Obviously the function in question is of exponential type unity. The polynomial $P_{2,n}$ reduces now to $\sin k\pi\sum_{i=0}^{n-1}(-k^2)^i{}'\beta_i(z)$. Applying Theorem 4 we see that the sequence $P_{2,n}(z)$ diverges for all z with the possible exception of integer multiples of π. On the other hand, if we consider the series $\sum_{n=1}^{\infty}{}'\alpha_n(z)/n$, it is seen by applying Theorem 4 that it converges uniformly for all z to an integral function of exponential type *at most* equal to unity. We shall show that the exponential type of the limit function is *at least* unity, and hence is actually equal to 1.

Denoting the limit function by $c(z)$, we first observe that

$$c^{(2n)}(0) = \frac{1}{n}, \quad n > 1.$$

Now the exponential type of a function may be expressed in terms of the derivatives of the function at a fixed point. Thus, if $f(z)=\sum c_n(z-h)^n$ is the Taylor series of $f(z)$ about $z=h$, the exponential type σ of $f(z)$ is given by

$$\sigma = \limsup_{n\to\infty} n\,|c_n|^{1/n}/e.\dagger \tag{35}$$

The limit of the right hand member of (35) for even n is now equal to unity; hence the exponential type of $c(z)$ is at least equal to unity.

Returning to the previous example $a=0$, $b=\pi$, $f(z)=\sin kz$ but allowing k to have any non-integer value, $|k|>1$, we conclude as above that the sequence $P_{2,n}$ diverges for all z that are not integer multiples of π. For the latter points $P_{2,n}(z)$ converges and to the respective values which the function in question takes on there. The proof of this follows at once if at these points we compute the remainder by means of (16), provided we utilize (20) with an appropriately large M.

We shall close Part II with a sample existence theorem that follows from the preceding work.

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* That is, $\pi/|a-b|$ is the greatest lower bound of values of k for which $e(z)=O(e^{k|z|})$ holds. In the theory of functions of a complex variable what we have termed a function of ***exponential type*** σ is known as a function of ***order*** 1 and ***type*** σ. See Bieberbach, loc. cit., pp. 227, 228.

† Bieberbach, loc. cit., p. 231. This formula is derived from the second italicized statement on that page, with the order equated to 1.

THEOREM 9. *If* $\sum_{n=0}^{\infty}(-1)^n c_n$ *converges, then there exists a function* $f(z)$ *satisfying the following conditions*:

(1) $f^{(2n)}(0)=c_n$;

(2) $f(z)$ *is of exponential type at most equal to* 1;

(3) $f(z)$ *is either odd about* π *or even about* $\pi/2$.

These conditions, moreover, determine $f(z)$ *uniquely except for an additive term* $D\sin z$, *where* D *is a constant.*

The proof of this theorem consists simply in considering the functions defined by the series $\sum_{n=0}^{\infty} c_n{}'\alpha_n(z)$, $\sum_{n=0}^{\infty} c_n[{}'\alpha_n(z)+{}'\beta_n(z)]$.

By expanding the function $f(z)$ in a Taylor series and rephrasing Theorem 9 in terms of the resulting coefficients, one obtains existence theorems for solutions of a proper infinite system of linear equations in an infinite number of variables. Proceeding in this way with the Taylor expansion of $\sum c_n \cdot[{}'\alpha_n(z)+{}'\beta_n(z)]$ about $z-(\pi/2)$:

$$\sum_{n=0}^{\infty} c_n[{}'\alpha_n(z)+{}'\beta_n(z)] = \sum_{n=0}^{\infty} x_n[z-(\pi/2)]^{2n}/(2n)!$$

we obtain, as an equivalent of Theorem 9 for the case in which $f(z)$ is even about $z-(\pi/2)$, the following existence theorem:

THEOREM 9a. *The system of equations*

$$\sum_{n=0}^{\infty} x_{n+m}(\pi/2)^{2n}/(2n+2m)! = c_m \qquad (m=0,1,2,\cdots),$$

where c_m, x_n *are subject to the conditions*

$$\sum_{m=0}^{\infty}(-1)^m c_m \text{ is convergent,}$$

$$\limsup_{n\to\infty} |x_n|^{1/n} \leqq 1,$$

respectively, possesses the solutions given by

$$x_n = 2\sum_{m=0}^{\infty} c_m{}'\alpha_{m+n}(\pi/2) + D(-1)^n$$

where D *is an arbitrary constant, and no others.*

That the condition x_n is the same as condition 2 of Theorem 9, follows by the use of (35).

PART III. THE POLYNOMIALS $P_{m,n}$ FOR $m>2$

12. Orientation of the problem. The situation that arises in connection with the polynomials $P_{m,n}(z)$ for $m>2$ has been briefly discussed in §3, and compared with the cases $m=2$. The sufficient condition for the convergence of the polynomials $P_{m,n}(z)$ for $m>2$ there described is proved below in Theorem 10 for the case $m=3$ and in a manner that requires no information regarding the distribution and nature of the characteristic values of the system (3_m), (4_m). Such is not the case, however, with regard to the more delicate questions of the broader sufficient conditions for special types of functions, analogous to those of Theorem 2, nor with regard to the discussion of the general nature of the convergence of the polynomials $P_{m,n}$ or of their equivalent series.

The characteristic values in question are shown to be roots of a certain transcendental function of λ^m, but little information concerning the location and multiplicity of the roots of this function seems available for $m>2$. Now as a certain amount of such information is needed for discussing the above questions, certain assumptions are made in §§16, 17 regarding the roots nearest the origin, assumptions that amount to confining oneself to the general case. Thus for $m=3$, let $\mu_1, \mu_2, \cdots$ be the roots in question, arranged in order of non-decreasing distance from the origin; if it is assumed that $|\mu_1|<|\mu_2|$ and that μ_1 is simple, then it is shown in Theorem 11a that the allowable exponential type of $f(z)$ may be enlarged to any number less than $|\mu_2|$, provided that $f(z)$ satisfy certain auxiliary conditions. Theorem 11b then shows that one may proceed even further in this direction. Except for the above assumptions which confine the discussion to the general case, these theorems form the complete analogue of the sufficient conditions of Part II for the case $m=2$. Likewise the treatment of the general convergence of the polynomials $P_{3,n}$ or of their equivalent series, given in Theorem 12, is as complete as the treatment of the analogous question for $m=2$, except that it is confined to an even more restricted general case. Yet it must be said that the generalization to cases beyond $m=2$ is by no means an obvious one, as the case $m=2$ is somewhat misleading in its simplicity, and that it required a considerable search to reveal the facts for higher values of m.

While only the case $m=3$ is treated at length, it is fairly typical of the general case of m beyond 2, and corresponding results for any m may be obtained in general by changing the order of matrices and the range of subscripts involved. Whatever features of this generalization are not obvious are discussed briefly at the end of Part III in §18.

13. The Green's functions for $m=3$. We start the treatment of the case

$m=3$ by considering the Green's functions $G_{3,i}(z, s; \lambda)$, $i=1, 2, 3$, mentioned at the end of §4; we shall denote them, however, by $G_i(z, s; \lambda)$. They are defined by means of the equations

$$(36) \qquad \frac{\partial^3 G_i(z, s; \lambda)}{\partial s^3} = -\lambda^3 G_i(z, s; \lambda),$$

$$(37_1) \qquad G_i(z, a_i; \lambda) = 0,$$

$$(37_2) \qquad \left.\frac{\partial G_i(z, s; \lambda)}{\partial s}\right|_{s=a_i} = 0,$$

$$(37_3) \qquad \sum_{i=1}^{3} G_i(z, z; \lambda) = 0,$$

$$(37_4) \qquad \left.\sum_{i=1}^{3} \frac{\partial}{\partial s} G_i(z, s; \lambda)\right|_{s=z} = 0,$$

$$(37_5) \qquad \left.\sum_{i=1}^{3} \frac{\partial^2}{\partial s^2} G_i(z, s; \lambda)\right|_{s=z} = 1.$$

For $\lambda \neq 0$ solutions of the first three equations above may be put in the form

$$G_i(z, s; \lambda) = \overline{G}_i s_3[\lambda(a_i - s)],$$

where $\overline{G}_i$ are independent of s, and s_3 stands for the power series

$$(38) \qquad \begin{aligned} s_3(x) &= \frac{x^2}{2!} + \frac{x^5}{5!} + \cdots + \frac{x^{3n+2}}{(3n+2)!} + \cdots \\ &= \frac{1}{3}(e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}), \quad \omega = e^{2\pi i/3}. \end{aligned}$$

Substituting this form of G_i in (37_3), (37_4), (37_5) we are led to the equations

$$\overline{G}_1 s_3[\lambda(a_1 - z)] + \overline{G}_2 s_3[\lambda(a_2 - z)] + \overline{G}_3 s_3[\lambda(a_3 - z)] = 0,$$

$$\overline{G}_1 \frac{\partial}{\partial z} s_3[\lambda(a_1 - z)] + \overline{G}_2 \frac{\partial}{\partial z} s_3[\lambda(a_2 - z)] + \overline{G}_3 \frac{\partial}{\partial z} s_3[\lambda(a_3 - z)] = 0,$$

$$\overline{G}_1 \frac{\partial^2}{\partial z^2} s_3[\lambda(a_1 - z)] + \overline{G}_2 \frac{\partial^2}{\partial z^2} s_3[\lambda(a_2 - z)] + \overline{G}_3 \frac{\partial^2}{\partial z^2} s_3[\lambda(a_3 - z)] = 1,$$

and solving them in the case the determinant D does not vanish, we obtain

$$(39) \qquad \begin{aligned} G_i(z, s; \lambda) &= \frac{s_3[\lambda(a_i - s)]}{D} \begin{vmatrix} s_3[\lambda(a_{i+1} - z)] & s_3[\lambda(a_{i+2} - z)] \\ \frac{\partial}{\partial z} s_3[\lambda(a_{i+1} - z)] & \frac{\partial}{\partial z} s_3[\lambda(a_{i+2} - z)] \end{vmatrix} \\ &= s_3[\lambda(a_i - s)] N_i(\lambda z)/D, \end{aligned}$$

where $a_i = a_j$ for $i \equiv j \pmod 3$, and N_i is the two-rowed determinant appearing in the second member above.

The determinant D is independent of z and s, and is an integral function of λ^3. This may be seen by replacing the functions s_3 in the determinant representation of D by the sum of exponentials from (38), whereupon the determinant may be factored thus:

$$D = -\frac{\lambda^3}{27}\begin{vmatrix} e^{\lambda(a_1-z)} & e^{\omega\lambda(a_1-z)} & e^{\omega^2\lambda(a_1-z)} \\ e^{\lambda(a_2-z)} & e^{\omega\lambda(a_2-z)} & e^{\omega^2\lambda(a_2-z)} \\ e^{\lambda(a_3-z)} & e^{\omega\lambda(a_3-z)} & e^{\omega^2\lambda(a_3-z)} \end{vmatrix} \cdot \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$(40) \qquad = i\lambda^3 3^{-3/2}\begin{vmatrix} e^{\lambda a_1} & e^{\lambda\omega a_1} & e^{\lambda\omega^2 a_1} \\ e^{\lambda a_2} & e^{\lambda\omega a_2} & e^{\lambda\omega^2 a_2} \\ e^{\lambda a_3} & e^{\lambda\omega a_3} & e^{\lambda\omega^2 a_3} \end{vmatrix}$$

$$= i\lambda^3 3^{-1/2}[s_3''(\lambda b) - s_3''(\lambda c)],$$

where

$$b = a_1 + \omega a_2 + \omega^2 a_3, \quad c = \omega^2 a_1 + \omega a_2 + a_3.$$

The integral function $D = D(\lambda)$ belongs to the type of function investigated by G. Pólya* and for which he proved the existence of an infinite number of roots and investigated the distribution of the roots at infinity. Now the roots, other than 0, of $D(\lambda)$, whose existence is thus assured, are precisely the set of values of λ for which non-trivial solutions of

$$(3_3) \qquad d^3u(z)/dz^3 - \lambda^3 u(z) = 0,$$

$$(4_3) \qquad u(a_1) = u(a_2) = u(a_3) = 0$$

exist. For, substituting an arbitrary solution of (3_3) in the form $A_1e^{\lambda z} + A_2e^{\lambda\omega z} + A_3e^{\lambda\omega^2 z}$ in (4_3), we obtain equations the determinant of whose coefficients is the same as the determinant in (40). It is apparent from (39) that these roots are (for general values of z and s) poles of the Green's functions G_i.

By employing for s_3 in (39) the power series (38) it is seen that $\lambda = 0$ is not a pole of G_i. The system (36), (37_i) is readily shown to possess a unique solution for $\lambda = 0$; this solution may be obtained from (39) by letting λ approach 0. The use of these power series also shows that for fixed s and z, G_i depend upon λ^3 only, as may also be inferred from (36), (37_i).

As already mentioned in §4, the name "Green's functions" for G_i is due to their connection with the differential equation

$$(3_3') \qquad d^3u(z)/dz^3 - \lambda^3 u(z) = -v(z)$$

* G. Pólya, *Geometrisches über die Verteilung der Nullstellen gewisser ganzer Funktionen*, Münchener Berichte, 1920, pp. 285–290.

and the boundary conditions (4_3). Thus, for values of λ not poles of G_i this system possesses the unique solution

$$u(z) = \sum_{i=1}^{3} \int_{z}^{a_i} G_i(z, s; \lambda) v(s) ds. \tag{6_3}$$

That the solution, if it exists, is unique, follows from the fact that the homogeneous system (3_3), (4_3) possesses as its only solution the solution $u(z)=0$. That (6_3) is a solution may be verified by differentiation and substitution, making use of the following properties of G_i:

$$\begin{aligned} \frac{\partial^3 G_i}{\partial z^3} &= \lambda^3 G_i, \\ G_i(a_j, s; \lambda) &= 0 \text{ for } i \neq j, \\ \sum_{i=1}^{3} G_i(z, z; \lambda) &= 0, \\ \sum_{i=1}^{3} \frac{\partial}{\partial z} G_i(z, s; \lambda) \Big|_{s=z} &= 0, \\ \sum_{i=1}^{3} \frac{\partial^2}{\partial z^2} G_i(z, s; \lambda) \Big|_{s=z} &= 1. \end{aligned} \tag{41}$$

The first two of these relations follow from (39) by differentiation and substitution; the remaining equations result from

$$\sum_{i=1}^{3} G_i(z, s; \lambda) = s_3[\lambda(z - s)], \tag{42}$$

an equation whose validity is manifest from (36), (37_3), (37_4), (37_5).

It follows from the above that the functions $G_i(z, s; \lambda)$ may be expanded in powers of λ^3:

$$G_i(z, s; \lambda) = \sum_{n=0}^{\infty} H_{i,n}(z, s) \lambda^{3n}, \tag{43}$$

these expansions being valid for $|\lambda| < \rho_3$, where ρ_3 is the absolute value of the roots of $D(\lambda)/\lambda^6$ nearest the origin (there are always at least three of them).

By substituting these power series in the various equations satisfied by G_i and equating coefficients of like powers of λ^3, one derives various properties of $H_{i,n}$. For a fixed n the latter may also be shown to be the Green's functions of the system $d^{3n}u(z)/dz^{3n} = v(z)$, $u^{(3j)}(a_i) = 0$; $i = 1, 2, 3$; $j = 0, 1, \cdots, n-1$. Finally, the functions $G_i(z, s; \lambda)$ may be shown to constitute the resolvent system to the kernel system $H_{i,0}(z, s)$. These statements are proved in a manner quite similar to the proof of the analogous statements about A, B, A_n, B_n in §§5, 6.

14. Expression of the polynomials $\alpha_{i,n}(z)$ and of the remainder $f(z)-P_{3,n}(z)$ in terms of $H_{i,n}(z, s)$. After the manner of §6 we now apply the formula

$$(7_3)\quad \int_{s_1}^{s_2}[u(s)v'''(s) + u'''(s)v(s)]ds = u(s)v''(s) - u'(s)v'(s) + u''(s)v(s)\Big]_{s_1}^{s_2}$$

to the pairs of functions

$$f(s),\ H_{i,0}(z, s);\ f'''(s),\ H_{i,1}(z, s);\ \cdots;\ f^{(3n)}(s),\ H_{i,n}(z, s),$$

where i has one of the values 1, 2, 3, between the limits z and a_i, and over the same path for all the $n+1$ integrations. Adding the equations resulting for a fixed i, and utilizing the relations

$$\frac{\partial^3 H_{i,n}(z, s)}{\partial s^3} = \begin{cases} -H_{i,n-1}(z, s) & \text{for } n > 0, \\ 0 & \text{for } n = 0 \end{cases}$$

which follow from (36), we see that the sum of the left hand members reduces to $\int_z^{a_i} f^{(3n+3)}(s)H_{i,n}(z, s)ds$. Adding the three equations thus obtained for $i=1, 2, 3$ and simplifying the right hand members by means of the various boundary value properties of $H_{i,n}$ which follow from (37_i) we are led to

$$f(z) - \sum_{i=1}^{3}\sum_{j=0}^{n} \frac{\partial^2 H_{i,j}(z, s)}{\partial s^2}\bigg|_{s=a_i} f^{(3j)}(a_i)$$
$$= -\sum_{i=1}^{3}\int_z^{a_i} H_{i,n}(z, s) f^{(3n+3)}(s)ds.$$

Now let $i=1, 2, 3$; $n=0, 1, \cdots$, and let $\alpha_{i,n}(z)$ be the polynomial of degree $3n+2$ whose derivatives of orders $3j$ $(j=0, 1, \cdots)$ all vanish at $z=a_1$, a_2, a_3 with exception of the $3n$th derivative at $z=a_i$, whose value is 1. The existence of these polynomials follows from equations $(1_{3,n+1})$. They are quite analogous to the polynomials $\alpha_n(z)$, $\beta_n(z)$ of Part II. We obviously have

$$P_{3,n}(z) = \sum_{j=0}^{n-1}\sum_{i=1}^{3} f^{(3j)}(a_i)\alpha_{i,j}(z).$$

If we put $f(z) = \alpha_{i,n}(z)$ in the formula derived above, we get

$$(44)\qquad \alpha_{i,n}(z) = \frac{\partial^2 H_{i,n}(z, s)}{\partial s^2}\bigg|_{s=a_i},$$

and therefore may write it in the form

$$f(z) - P_{3,n+1}(z) = -\sum_{i=1}^{3} \int_{z}^{a_i} H_{i,n}(z, s) f^{(3n+3)}(s) ds. \tag{45}$$

This "remainder" formula is the complete analogue of (16), and will form the basis of the proof of Theorem 10 in the next section.

From (39), (43), (44) follows the validity of

$$\sum_{n=0}^{\infty} \alpha_{i,n}(z)\lambda^{3n} = \left.\frac{\partial^2 G_i(z, s; \lambda)}{\partial s^2}\right|_{s=a_i} = \frac{\lambda^2 N_i(\lambda z)}{D(\lambda)}, \quad |\lambda| < \rho_3. \tag{46}$$

15. **Sufficient conditions for convergence of $P_{3,n}(z)$ to $f(z)$.** We shall now prove

THEOREM 10. *If $f(z)$ is an integral function, such that*

$$f(z) = O(e^{k|z|}), \quad k < \rho_3, \tag{47}$$

where ρ_3 is the absolute value of the roots of $D(\lambda)/\lambda^6$ that are nearest the origin, then $P_{3,n}(z)$ converges to $f(z)$, the convergence being uniform in any finite region of the z-plane.

We recall that $D(\lambda)/\lambda^6$ is an integral function of λ^3 (see (40)) whose roots coincide with the values of λ for which non-trivial solutions of (3_3), (4_3) exist. Hence the conditions of this theorem are equivalent to those of §3.

The proof of this theorem is quite similar to that of Theorem 1, and consists in showing that the remainder given by the right hand member of (45) approaches 0 as n becomes infinite.

First we recall that from (47) follows the inequality

$$|f^{(n)}(z)| < Cn^{1/2}k^n, \tag{24}$$

where z ranges over any finite region, and C is a constant depending upon that region (for proof of (24) see §8). We then proceed to estimate $H_{i,n}(z, s)$ for large n.

An estimate of $H_{i,n}$ which is not as precise as the one obtained for A_n, B_n in §5, but is nevertheless sufficient for the purpose at hand, may be obtained from the fact that the generating functions $G_i(z, s; \lambda)$ (see (43)) are analytic in the λ-plane inside a circle of radius ρ_3 and center at the origin, and for z and s in arbitrary finite regions of their planes. We have then for z and s as stated, and $|\lambda| = k'$, where k' lies between ρ_3 and the constant k for which (47) holds,

$$|G_i(z, s; \lambda)| < M,$$

where M is a proper constant. Hence by applying Cauchy's integral

$$|H_{i,n}(z, s)| < Mk'^{-3n}.$$

Combining this inequality with (24) in which n has been replaced by $3n+3$, we see that by choosing n large enough the integrands in the right hand member of (45) can be made uniformly small in absolute value for z and s in finite arbitrary regions. The proof of Theorem 10 is thus complete.

We shall now show that Theorem 10 will not hold if in (47) we replace ρ_3 by any larger value. Let λ_1, $|\lambda_1|=\rho_3$, be one of the roots of $D(\lambda)/\lambda^6$ nearest the origin. As explained in §13, there will exist constants C_1, C_2, C_3, not all zero, and such that

$$u(z) = C_1e^{\lambda_1 z} + C_2e^{\omega\lambda_1 z} + C_3e^{\omega^2\lambda_1 z}$$

vanishes at $z=a_1$, $z=a_2$, $z=a_3$. The function $u(z)$ obviously satisfies

$$u(z) = O(e^{k|z|})$$

for any constant k greater than $|\lambda_1|=\rho_3$, but the polynomials $P_{3,n}$ formed for $u(z)$ vanish identically. From Theorem 10 it now follows that $u(z)$ must fail to satisfy (47).

16. **Extension of the sufficient conditions.** While for arbitrary functions $f(z)$, the exponential type conditions of the preceding section are the best possible ones of their type, by restricting $f(z)$ properly one may replace the exponential type conditions by more lenient ones. This is analogous to the situation which obtains for $P_{2,n}(z)$ as portrayed in Theorems 1 and 2.

From the original representation of $D(\lambda)$ as the determinant

$$|\,\partial^j s_3[\lambda(a_i - z)]/\partial z^j\,|$$

(see §13), it follows that D is the Wronskian of the three solutions of $d^3u(z)/dz^3+\lambda^3u(z)=0$,

$$s_3[\lambda(a_1 - z)],\quad s_3[\lambda(a_2 - z)],\quad s_3[\lambda(a_3 - z)].$$

Hence the roots of $D(\lambda)=0$ are precisely the values of λ for which the above three functions are linearly dependent:

$$D_1s_3[\lambda_n(a_1 - z)] + D_2s_3[\lambda_n(a_2 - z)] + D_3s_3[\lambda_n(a_3 - z)] \equiv 0, \tag{48}$$

where λ_n is a root of $D(\lambda)$ and D_1, D_2, D_3 are constants, not all zero. Differentiating this equation with respect to z, we may solve it and the resulting equation for the ratios of the two-rowed determinants:

$$\begin{vmatrix} s_3[\lambda_n(a_{i+1} - z)] & s_3[\lambda_n(a_{i+2} - z)] \\ \dfrac{\partial}{\partial z} s_3[\lambda_n(a_{i+1} - z)] & \dfrac{\partial}{\partial z} s_3[\lambda_n(a_{i+2} - z)] \end{vmatrix} = N_i(\lambda_n z)$$

(see (39)), and get

$$N_1(\lambda_n z): N_2(\lambda_n z): N_3(\lambda_n z) = D_1: D_2: D_3. \tag{49}$$

It may be shown by expressing s_3 in form of exponentials that no two of the functions $s_3[\lambda(a_i-z)]$ are ever linearly dependent. Therefore, no one of the above constants D_i vanishes.

Again, no one of the functions $N_i(\lambda z)$ ever reduces to zero identically, as this would imply the linear dependence of $s_3[(a_{i+1}-z)]$, $s_3[\lambda(a_{i+2}-z)]$. Hence it follows from (49) that any two of the functions $N_i(\lambda_n z)$ are linearly dependent.

We shall suppose now that in the λ^3-plane $D(\lambda)/\lambda^6$ possesses only one root, λ_1^3, which is nearest the origin, and that this root is simple. The principal part of G_i at λ_1^3 is

$$\frac{s_3[\lambda_1(a_i - s)]N_i(\lambda_1 z)}{(\lambda^3 - \lambda_1^3)(dD/d\lambda^3 \mid_{\lambda^3=\lambda_1^3})};$$

we shall denote the numerator of this fraction by $G_i^{\lambda_1}(z, s)$ and $dD/d\lambda^3 \mid_{\lambda^3=\lambda_1^3}$ by C_1.

If we expand G_i in a Laurent series about λ_1^3, substitute in equations (36), (37_i) and equate coefficients of $(\lambda^3-\lambda_1^3)^{-1}$ on both sides, we find that $G_i^{\lambda_1}$ satisfy the differential equations

$$\frac{\partial^3 G_i^{\lambda_1}}{\partial s^3} = -\lambda_1^3 G_i^{\lambda_1} \tag{50}$$

and homogeneous boundary conditions of the type (37_i).

From (50) and the last three boundary conditions, follows

$$\sum_{i=1}^{3} G_i^{\lambda_1}(z, s) = \sum_{i=1}^{3} s_3[\lambda_1(a_i - s)]N_i(\lambda_1 z) = 0 \tag{51}$$

identically in z and s, a relation which is also easily inferred from (48) and (49).

With these preliminaries disposed of, we shall now prove

THEOREM 11a. *Suppose that* $D(\lambda)/\lambda^6$ *possesses in the* λ^3*-plane a single root,* λ_1^3, *which is nearest the origin, and which is simple, and let* λ_2^3 *be the root of next greater absolute value. Let* D_1, D_2, D_3 *be the constants for which* (49) *holds for* $\lambda=\lambda_1$.

Sufficient conditions in order that $P_{3,n}(z)$ *approach* $f(z)$ *as* n *becomes infinite, and uniformly for* z *in any finite region, are that*

(1) $f(z)$ *is integral and satisfies*

$$f(z) = O(e^{k|z|}),\ k < |\lambda_2|;$$

(2) $$\sum_{i=1}^{3} D_i f^{(3n)}(a_i) = 0 \qquad (n = 0, 1, 2, \cdots);$$

(3) $$D_2 \int_{a_1}^{a_2} s_3[\lambda_1(a_2 - s)]f(s)ds + D_3 \int_{a_1}^{a_3} s_3[\lambda_1(a_3 - s)]f(s)ds = 0.$$

The last condition is equivalent to

$$\sum_{i=1}^{3} D_i \int_{z}^{a_i} s_3[\lambda_1(a_i - s)]f(s)ds = 0.$$

First, as regards condition 3, consider

$$\sum_{i=1}^{3} \int_{z}^{a_i} G_i^{\lambda_1}(z, s)f(s)ds = \sum_{i=1}^{3} \int_{z}^{a_1} G_i^{\lambda_1}(z, s)f(s)ds$$
$$+ \int_{a_1}^{a_2} G_2^{\lambda_1}(z, s)f(s)ds + \int_{a_1}^{a_3} G_3^{\lambda_1}(z, s)f(s)ds.$$

Using (51) and (49) (and since none of the constants D_i vanishes) we obtain for the right hand member above

$$\sum_{i=1}^{3} \int_{z}^{a_i} G_i^{\lambda_1}(z, s)f(s)ds = N_2(\lambda_1 z)\left(\int_{a_1}^{a_2} s_3[\lambda_1(a_2 - s)]f(s)ds \right.$$
$$\left. + \frac{D_3}{D_2} \int_{a_1}^{a_3} s_3[\lambda_1(a_2 - s)]f(s)ds \right).$$

On account of the first form of condition 3, the sum in the last parenthesis vanishes. Hence

$$\sum_{i=1}^{3} \int_{z}^{a_i} G_i^{\lambda_1}(z, s)f(s)ds = 0$$

identically in z, and replacing $G_i^{\lambda_1}$ by $N_i(\lambda_1 z)\ s_3[\lambda_1(a_i - s)]$ and utilizing (49), the second form of condition 3 follows. Conversely, by putting $z = a_1$ in the latter form of the condition, the first form is obtained.

We next proceed with the proof by recalling the remainder formula

(45) $$f(z) - P_{3,n}(z) = - \sum_{i=1}^{3} \int_{z}^{a_i} H_{i,n}(z, s) f^{(3n+3)}(s)ds;$$

here $H_{i,n}$ is defined by

(43) $$G_i(z, s; \lambda) = \sum_{n=0}^{\infty} H_{i,n}(z, s)\lambda^{3n}.$$

Break up G_i into the sum of its principal part at λ_1 and a function which is analytic for $|\lambda|<\lambda_2$; let $\sum_{n=0}^{\infty} H_{i,n}^{\lambda_1}(z, s)\lambda^{3n}$ be the expansion of the latter component of G_i. We have

$$H_{i,n}(z, s) = H_{i,n}^{\lambda_1}(z, s) + [G_i^{\lambda_1}(z, s)/C_1(-\lambda_1)^{3n+3}].$$

Now break up the remainder in (45) into R_1+R_2, where

$$R_1 = \sum_{i=1}^{3} \int_z^{a_i} H_{i,n}^{\lambda_1}(z, s) f^{(3n+3)}(s)\,ds,$$

$$C_1R_2 = \sum_{i=1}^{3} \int_z^{a_i} G_i^{\lambda_1}(z, s) f^{(3n+3)}(s)\,ds/(-\lambda_1)^{3n+3}.$$

Now,

$$H_{i,n}^{\lambda_1}(z, s) = O[(|\lambda_2| - \epsilon)^{-3n}],\ \epsilon > 0,$$

since $\sum_{n=0}^{\infty} H_{i,n}^{\lambda_1}(z, s)\lambda^{3n}$ is analytic for $|\lambda| < |\lambda_2|$. From this and condition 1 of Theorem 11a, one may prove, by the same methods as were used for Theorems 1 to 10, that R_1 approaches zero as n becomes infinite. We shall now show that R_2 vanishes for any n.

To this end apply the same procedure that was employed in §14 to pairs of functions $f(s)$, $H_{i,0}(z, s)$; $f'''(s)$, $H_{i,1}(z, s)$; $\cdots$ to the pairs of functions

$$f(s), \frac{G_i^{\lambda_1}(z, s)}{(-\lambda_1)^3};\ f'''(s), \frac{G_i^{\lambda_1}(z, s)}{(-\lambda_1)^6};\ \cdots;\ f^{(3n)}(s), \frac{G_i^{\lambda_1}(z, s)}{(-\lambda_1)^{3n+3}}.$$

On utilizing equation (50) and the homogeneous boundary conditions of type (37_i) satisfied by $G_i^{\lambda_1}$, we obtain a formula analogous to (44):

$$\sum_{i=1}^{3} \int_z^{a_i} f(s) G_i^{\lambda_1}(z, s)\,ds - \int_z^{a_i} f^{(3n+3)}(s) \frac{G_i^{\lambda_1}(z, s)}{(-\lambda_1)^{3n+3}}\,ds$$
$$= \sum_{i=1}^{3} \sum_{j=0}^{n} f^{(3j)}(s) \frac{\partial^2 G_i^{\lambda_1}(z, s)}{(-\lambda_1^{3+3j})\partial s^2}\Bigg|_{s=a_i}.$$

Now $\sum_{i=1}^{3} \int_z^{a_i} G_i^{\lambda_1}(z, s) f(s)\,ds$ has been shown to vanish as a consequence of condition 3. The right hand sum, on replacing $G_i^{\lambda_1}(z, s)$ by $s_3[\lambda_1(a_i-s)] \cdot N_i(\lambda z)$ and carrying out the differentiations, is transformed into

$$\lambda_1^2 \sum_{j=0}^{n} \left(\sum_{i=1}^{3} f^{(3j)}(a_i) N_i(\lambda z)/(-\lambda_1)^{3+3j} \right).$$

Finally, utilizing (49) and condition 2, one proves that for any j, the expression in the parentheses vanishes. The proof is then complete.

Theorem 11a is not the complete analogue of Theorem 2; by supposing that $f(z)$ in addition to satisfying the conditions of Theorem 11, also satisfies proper further conditions, one may still further extend the exponential type of $f(z)$ and still have $P_{3,n}$ converge to $f(z)$. In this connection we have

THEOREM 11b. *Let the roots of $D(\lambda)/\lambda^6$ in the λ^3-plane arranged in order of non-decreasing amplitude be*

$$\lambda_1^3,\ \lambda_2^3,\ \lambda_3^3,\ \cdots$$

and suppose that $|\lambda_1| < |\lambda_2| < |\lambda_3|$, and that λ_1 and λ_2 are simple roots. Let D_1, D_2, D_3; D_1', D_2', D_3' be the constants for which (49) *holds for $\lambda = \lambda_1$, $\lambda = \lambda_2$, respectively. Sufficient conditions in order that $P_{3,n}(z)$ approach $f(z)$ as n becomes infinite, and uniformly for z in any region, are that*

(1) *$f(z)$ is integral and satisfies*

$$f(z) = O(e^{k|z|}), \quad k < |\lambda_3|\ ;$$

(2)
$$\sum_{i=1}^{3} D_i f^{(3n)}(a_i) = 0,$$

$$\sum_{i=1}^{3} D_i' f^{(3n)}(a_i) = 0 \qquad (n = 0, 1, \cdots);$$

(3)
$$\sum_{i=1}^{3} D_i \int_z^{a_i} s_3[\lambda_1(a_i - s)] f(s) ds = 0,$$

$$\sum_{i=1}^{3} D_i' \int_z^{a_i} s_3[\lambda_2(a_i - s)] f(s) ds = 0.$$

It will be seen that the conditions of Theorem 11b include those of Theorem 11a. The proof of this theorem follows along similar lines by breaking up G_i into a sum of the principal parts at both λ_1^3 and λ_2^3, and a function which is analytic for $\lambda^3 < |\lambda_3|^3$, and correspondingly breaking up the coefficients $H_{i,n}$.

The matrix

$$\left\| \begin{matrix} D_1 & D_2 & D_3 \\ D_1' & D_2' & D_3' \end{matrix} \right\|$$

may be supposed to be of rank two. If it should ever happen that (for proper a_i) it is of rank 1, then the latter part of condition 2 is superfluous, and one could even further extend the permissible exponential type of $f(z)$.

17. **Convergence of the series $\sum_{n=0}^{\infty}\sum_{i=1}^{3} C_{i,n}\alpha_{i,n}(z)$.** We shall now prove the following analogue of Theorems 4–6:

THEOREM 12. *Let the roots of* $D(\lambda)/\lambda^6$ *in the* λ^3*-plane, arranged in order of non-decreasing amplitude, be*

$$\lambda_1^3, \lambda_2^3, \cdots,$$

and suppose that

$$|\lambda_1| < |\lambda_2| < |\lambda_3| < |\lambda_4|$$

and that $\lambda_1^3, \lambda_2^3, \lambda_3^3$ *are all simple roots. Write the equations* (49) *corresponding to the roots* λ_i, $i=1, 2, 3$, *in the form*

$$N_1(\lambda_i z):N_2(\lambda_i z):N_3(\lambda_i z) = D_{i,1}:D_{i,2}:D_{i,3}, \qquad D_{i,j} \neq 0.$$

Suppose that the determinant $|D_{i,j}|$ *does not vanish, and let* $\delta_{i,j}$ *be the reciprocal matrix to the matrix* $D_{i,j}$. *In order that the series*

$$\sum_{n=0}^{\infty}[C_{1,n}\alpha_{1,n}(z) + C_{2,n}\alpha_{2,n}(z) + C_{3,n}\alpha_{3,n}(z)] \tag{52}$$

converge for general values of z, *it is necessary that the three series*

$$\sum_{n=0}^{\infty} E_{i,n}/\lambda_i^{3n}; \quad E_{i,n} = \sum_{j=1}^{3} C_{j,n}D_{i,j} \qquad (i = 1, 2, 3), \tag{53}$$

all converge. Conversely, when these series are convergent, the series (52) *converges for all* z, *uniformly in any finite region, and may be broken up into a sum of the three series*

$$\sum_{n=0}^{\infty} E_{i,n}\beta_{i,n}(z);\ \beta_{i,n}(z) = \sum_{j=1}^{3}\delta_{j,i}\alpha_{j,n}(z) \qquad (i = 1, 2, 3), \tag{54}$$

convergent likewise and to integral functions $f_i(z)$ *of exponential type at most equal to* $|\lambda_i|$ *respectively, and satisfying the conditions*

$$\sum_{j=1}^{3} D_{k,j} f_i^{(3n)}(a_j) = 0 \text{ for } i \neq k. \tag{55}$$

It will be noticed that the latter conditions are of the same type as conditions 2 of Theorems 11a, 11b.

The constants $D_{i,j}$ may be definitely fixed by assigning their values for some one j. We shall suppose that $D_{i,1}=1$. If further we denote $N_1(\lambda z)$ by $N(\lambda z)$, we may write the equations which define $D_{i,j}$ in the form

$$N_i(\lambda_j z) = D_{j,i} N(\lambda_j z).$$

The statement of the necessary conditions in the theorem includes the somewhat vague phrase "converge for *general values of z*." The proof presently to be given will show that the series (53) converge if (52) converges for three values of z: z_1, z_2, z_3, such that the determinant $|N(\lambda_i z_j)|$ does not vanish. Now, as the theorem further states that the convergence of (53) insures the convergence of (52) for all z, it follows that the set of values of z for which (52) converges is such that for any three of its points, z_1, z_2, z_3, the equation $|N(\lambda_i z_j)| = 0$ holds. Hence this set, when it does not consist of the complete complex plane, is a discrete set with infinity as its only possible limiting point.

The proof of Theorem 12 is quite analogous to the proof of Theorems 4–6. The separation of the series (52) into the three series (54) is analogous to the breaking up of (31) into (33_i) (§10). Barring questions of convergence, the equivalence of (52) and the sum of (54) is manifest from the identity

$$\sum_{j=1}^{3} C_{j,n}\alpha_{j,n}(z) = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} C_{i,n}D_{j,i}\right)\left[\sum_{k=1}^{3}\delta_{k,j}\alpha_{k,n}(z)\right] = \sum_{j=1}^{3} E_{j,n}\beta_{j,n}(z),$$

an identity which itself follows from the relations between the elements of the mutually reciprocal matrices $D_{i,j}$, $\delta_{i,j}$:

$$\sum_{j=1}^{3} D_{j,i}\delta_{k,j} = \begin{cases} 1 \text{ for } i = k, \\ 0 \text{ for } i \neq k. \end{cases}$$

We shall now establish asymptotic formulas for the functions $\beta_{i,n}(z)$, based upon the fact that their generating functions possess in the λ^3-plane only one pole, namely, $\lambda^3 = \lambda_i^3$, inside the circle of radius $|\lambda_4^3|$. The generating functions of $\alpha_{i,n}(z)$ are given by

$$(46) \qquad \left.\frac{\partial^2 G_i(z, s; \lambda)}{\partial s^2}\right|_{s=a_i} = \frac{\lambda^2 N_i(z\lambda)}{D(\lambda)} = \sum_{n=0}^{\infty}\alpha_{i,n}(z)\lambda^{3n}.$$

From these equations, and since the principal part of $\lambda^2 N_i(z\lambda)/D(\lambda)$ at a simple root λ_j is

$$\lambda_j^2 N_i(z\lambda_j)/[C_j(\lambda^3 - \lambda_j^3)],$$

where

$$C_j = dD(\lambda)/d\lambda^3 \Big|_{\lambda^3 = \lambda_j^3},$$

follows

$$\alpha_{i,n}(z) = -\sum_{j=1}^{3} N_i(z\lambda_j)/(C_j\lambda_j^{3n+1}) + O(|\lambda_4| - \epsilon)^{-3n}$$

$$= -\sum_{j=1}^{3} N(z\lambda_j)D_{j,i}/(C_j\lambda_j^{3n+1}) + O(|\lambda_4| - \epsilon)^{-3n}, \ \epsilon > 0.$$

Hence

$$\begin{aligned}\beta_{i,n}(z) &= \sum_{j=1}^{3}\delta_{j,i}\alpha_{j,n}(z)\\ &= -\sum_{j=1}^{3}\delta_{j,i}\sum_{k=1}^{3}N(z\lambda_k)D_{k,j}/(C_k\lambda_k^{3n+1}) + O[(\,|\lambda_4| - \epsilon)^{-3n}].\end{aligned}$$

Carrying out the j-summation first and replacing $\sum_{j=1}\delta_{j,i}D_{k,j}$ by 1 or 0 according as $i = k$ or $i \neq k$, we obtain

$$\beta_{i,n}(z) = -N(z\lambda_i)/(C_i\lambda_i^{3n+1}) + O[(\,|\lambda_4| - \epsilon)^{-3n}].$$

Now suppose that the series (52) or, what amounts to the same thing, the series

$$\sum_{n=0}^{\infty}\left(\sum_{i=1}^{3}E_{i,n}\beta_{i,n}(z)\right), \tag{56}$$

converges for $z = z_1, z_2, z_3$, where these values are such that the determinant $|N(\lambda_i z_j)|$ does not vanish. Using the asymptotic representation developed for $\beta_{i,n}$ we get

$$\begin{aligned}\sum_{i=1}^{3}E_{i,n}\beta_{i,n}(z) &= -\sum_{i=1}^{3}E_{i,n}\{[N(z\lambda_i)/(C_i\lambda_i^{3n+1})]\\ &\quad + O[(\,|\lambda_4| - \epsilon)^{-3n}]\} = \epsilon(z, n),\end{aligned} \tag{57}$$

and conclude that $\epsilon(z, n)$ approaches zero for $z = z_1, z_2, z_3$ as n becomes infinite. Regarding the three equations thus obtained as linear equations in $E_{i,n}/\lambda_i^{3n}$, we find that the coefficients of these quantities approach the terms of the matrix $-N(z_j\lambda_i)/(C_i\lambda_i)$ as n becomes infinite. For sufficiently large n we may therefore solve for $E_{i,n}/\lambda_i^{3n}$ from these equations, and conclude that these quantities approach zero as n becomes infinite, and are therefore bounded in n. Hence when the asymptotic representations (57) are substituted in (56), part of the resulting series consisting of the O-terms converges for all z. Therefore the remaining part of the series (54), namely,

$$-\sum_{n=0}^{\infty}\left[\sum_{i=1}^{3}E_{i,n}N(z\lambda_i)/(C_i\lambda_i^{3n+1})\right],$$

converges for $z = z_1, z_2, z_3$. Finally, multiplying these last three convergent series by the terms of the various columns of the matrix which is reciprocal to $N(z_j\lambda_i)/(C_i\lambda_i)$ and changing signs, we obtain for the left hand members the series (53). These series consequently must converge.

Conversely, let the series (53) converge. By applying a proof similar to that of Theorem 4, and the above asymptotic representations for $\beta_{i,n}(z)$ one shows that the series (54) will converge for all z, and uniformly in any finite region. The generating function of $\beta_{i,n}(z)$,

$$\sum_{j=1}^{3}\delta_{j,i}\partial^2 G_j(z, s; \lambda)/\partial s^2\Big|_{s=a_j} = \sum_{n=0}^{\infty}\beta_{i,n}(z)\lambda^{3n},$$

is a solution of

$$\partial^3(\quad)/\partial z^3 = \lambda^3(\quad), \tag{41}$$

analytic in the λ^3-plane for $|\lambda^3| < |\lambda_4^3|$ with the exception of a pole of first order at $\lambda^3 = \lambda_i^3$ with residue $N(z\lambda_i)/C_i$. Likewise the latter function satisfies the equation (41) with λ^3 replaced by λ_i^3 and the partial derivative by a total derivative. These facts are established in the same manner as the initial equations (41), and from the above asymptotic formulas for $\beta_{i,n}$. By utilizing them and proceeding as in Theorem 4, one may express the finite series in (54), $\sum_{n=0}^{N} E_{i,n}\beta_{i,n}(z)$, first as a contour integral around the origin, then around a circle between $|\lambda^3| = |\lambda_3^3|$ and $|\lambda^3| = |\lambda_4^3|$, and prove that the limit of (54) is a function $f_i(z)$ of exponential type at most equal to $|\lambda_i|$.

Finally, the proof of (55) may be carried out by showing that (55) is satisfied by each term of (54). This follows in a direct manner by differentiation and substitution provided it is recalled that $\beta_{i,n}^{(3m)}(a_k) = 0$ except for $m = n$ and $i = k$, in which case the value of the derivative is unity, and use is made of the relation $\sum_{j=1}^{3} D_{k,j}\delta_{j,i} = 0$ for $i \neq k$.

By applying this theorem to the case where $C_{i,n}$ are chosen as $f^{(3n)}(a_i)$ one may obtain necessary conditions in order that $P_{3,n}(z)$ converge to $f(z)$, and in this way formulate a theorem which is analogous to Theorem 7. Thus far we have not succeeded in proving what may be suspected to be the analogue of Theorem 8 to the effect that when the necessary conditions just mentioned are satisfied, $f(z)$ will differ from $\lim_{n\to\infty} P_{3,n}(z)$ by a linear combination of $N(\lambda_1 z)$, $N(\lambda_2 z)$, $N(\lambda_3 z)$.

18. **The polynomials $P_{m,n}(z)$ for $m = 1$ and for $m > 3$.** Sufficient conditions for the convergence of $P_{m,n}(z)$ to $f(z)$ as n becomes infinite, for the general case $m > 2$, have been outlined in §3. The proof of the sufficiency of these conditions for an arbitrary $m > 3$ may be carried out along the lines of Theorem 10 by means of Green's functions defined by a system of equations whose formation is obvious from (8), (9_i); (36), (37_i); it consists of the differential equation

$$\frac{\partial^3}{\partial s^3}(\quad) = (-\lambda)^m(\quad)$$

and of proper boundary conditions. The solution of this system can readily be expressed in terms of the function

$$s_m(x) = \frac{x^{m-1}}{(m-1)!} + \frac{x^{2m-1}}{(2m-1)!} + \frac{x^{3m-1}}{(3m-1)!} + \cdots.$$

Likewise, by making proper assumptions concerning the $m+1$ poles of the Green's functions, that are nearest the origin of the λ^m-plane, we may prove results analogous to those of §§16, 17.

In their severest form these assumptions are that these poles are simple (that is, of order 1), that no two of them are equally distant from the origin, and that none of the constants analogous to the constants D_i in equation (48) vanish.

As stated in §4, the nature of the convergence of the polynomials $P_{1,n}(z)$ is radically different from the convergence of $P_{m,n}(z)$ for $m>1$, since the polynomial $P_{1,n}(z)$ agrees with the first n terms of the Taylor expansion of $f(z)$ about $z=a_1$. It is of interest therefore to see what becomes of the Green's function and of the method of proof employed for $m>1$.

The solution of

$$(3_1') \qquad du(z)/dz - \lambda u(z) = -v(z)$$

satisfying

$$(4_1) \qquad u(a_1) = 0$$

is given by

$$(6_1) \qquad u(z) = \int_z^{a_1} e^{\lambda(z-s)} v(s)\,ds,$$

so that the Green's function is now given by

$$G(z, z; \lambda) = e^{\lambda(z-s)},$$

and it could therefore be defined after the manner of (8), (9_i); (36), (37_i) by means of the system

$$\frac{\partial G(z, s; \lambda)}{\partial s} = -\lambda G(z, s; \lambda),$$

$$G(z, z; \lambda) = 1.$$

Successive application of the formula

$$(7_1) \qquad \int_{s_1}^{s_2} [u(s)v'(s) + u'(s)v(s)]\,ds = u(s)v(s)\Big]_{s_1}^{s_2}$$

to the derivatives of $u(s)$ and the coefficients resulting from the expansion of G in powers of λ and between the limits z and a_1 leads to the familiar form for Taylor's series with a remainder:

$$u(z) = \sum_{i=0}^{n} u^{(i)}(a_1)(z - a_1)^i/i! - \int_z^a (z - s)^n u^{(n+1)}(s)ds/n!;$$

it is generally obtained by successive integrations by parts of $\int_z^a u'(s)ds$.

As stated in the introduction (§4), "the reason" for the difference in the convergence of $P_{1,n}(z)$, and of $P_{m,n}(z)$ for $m>1$, is due to the fact that the Green's function in the former case is integral in the parameter (rather than meromorphic); hence the coefficients resulting from its expansion in powers of λ possess a different asymptotic behavior that now allows the remainder to approach 0 for a much wider class of functions.

It is easy to give examples of a system consisting of a non-homogeneous differential equation of *arbitrary* order and of proper boundary conditions, and for which the Green's functions are integral in the parameter. Thus, the differential equation $(3)'_m$ combined with the boundary conditions

$$u(a) = u'(a) = \cdots = u^{(m-1)}(a) = 0$$

has the solution

$$u(z) = \int_z^a s_m[\lambda(z - s)]v(s)ds,$$

where s_m is the integral function above defined. This system leads essentially to Taylor's expansion with the terms grouped in bunches of m each. Hence the special features displayed by the polynomials $P_{1,n}$ are not due entirely to the fact that they are connected with a differential system of the *first* order. It will also be shown in §21 that differential systems of the first order may lead to polynomial approximations whose behavior is analogous to that of $P_{m,n}(z)$ for $m>1$.

Part IV. Divers expansions

19. Expansions suggested by the Taylor expansion of the Green's functions of Part II about an arbitrary value of the parameter. We saw in Part II that the approximations by means of $P_{2,n}(z)$ were intimately connected with the expansions of the Green's functions $A(z, s; \lambda)$, $B(z, s; \lambda)$ in powers of λ^2. We shall now consider the expansions of these Green's functions in powers of $\lambda^2-\lambda_0^2$, where λ_0 is an arbitrary constant, not a pole of A, B, and different from zero:

$$A(z, s; \lambda) = \sum_{n=0}^{\infty} A_{\lambda_0,n}(z, s)(\lambda^2 - \lambda_0^2)^n, \tag{58}$$

$$B(z, s; \lambda) = \sum_{n=0}^{\infty} B_{\lambda_0,n}(z, s)(\lambda^2 - \lambda_0^2)^n.$$

It will be found that these Taylor series suggest approximations to an analytic function by means of solutions of

(59) $$(D^2 - \lambda_0^2)^n = 0 \qquad (n = 1, 2, \cdots),$$

satisfying the same boundary conditions at a and b as were satisfied by $P_{2,n}(z)$; in (59) D^2 stands for the second derivative, while $(D^2-\lambda_0^2)^n$ stands for n successive applications of the operator $(D^2-\lambda_0^2)$.

The expansions (58) are obviously valid in the λ^2-plane inside a circle with center at λ_0^2 and passing through the nearest pole (or poles) of A, B, that is, for

$$|\lambda^2 - \lambda_0^2| < |\lambda_1^2 - \lambda_0^2|,$$

where λ_1^2 denotes that nearest pole (or either of the two nearest poles in case there are two of them). Substituting (58) in (8) written in the form

$$\left[\left(\frac{\partial^2}{\partial s^2} - \lambda_0^2\right) - (\lambda^2 - \lambda_0^2)\right](A,B) = 0,$$

as well as in (9_i), and equating coefficients of like powers of $(\lambda^2-\lambda_0^2)$ on both sides of the resulting equations, we find that $A_{\lambda_0,n}(z, s)$, $B_{\lambda_0,n}(z, s)$ satisfy the differential equations

(60)
$$\left(\frac{\partial^2}{\partial s^2} - \lambda_0^2\right)A_{\lambda_0,n}(z, s) = \begin{cases} 0 \text{ for } n = 0, \\ A_{\lambda_0,n-1}(z, s) \text{ for } n > 0, \end{cases}$$
$$\left(\frac{\partial^2}{\partial s^2} - \lambda_0^2\right)B_{\lambda_0,n}(z, s) = \begin{cases} 0 \text{ for } n = 0, \\ B_{\lambda_0,n-1}(z, s) \text{ for } n > 0, \end{cases}$$

as well as the same boundary conditions (13_i) as were satisfied by A_n, B_n.

Next consider the expansions of

$$\partial A(z, s; \lambda)/\partial s\Big|_{s=a}, \quad \partial B(z, s; \lambda)/\partial s\Big|_{s=a} \quad \text{in powers of } \lambda^2 - \lambda_0^2;$$

(61)
$$\left.\frac{\partial A(z, s; \lambda)}{\partial s}\right|_{s=a} = \frac{\sinh \lambda(z - b)}{\sinh \lambda(a - b)} = \sum_{n=0}^{\infty} \alpha_{\lambda_0,n}(z)(\lambda^2 - \lambda_0^2)^n,$$
$$\left.\frac{\partial B(z, s; \lambda)}{\partial s}\right|_{s=b} = \frac{\sinh \lambda(z - a)}{\sinh \lambda(a - b)} = \sum_{n=0}^{\infty} \beta_{\lambda_0,n}(z)(\lambda^2 - \lambda_0^2)^n.$$

The coefficients $\alpha_{\lambda_0,n}(z)$, $\beta_{\lambda_0,n}(z)$ are obviously related to the coefficients of the expansions in (58) as follows:

$$\alpha_{\lambda_0,n}(z) = \partial A_{\lambda_0,n}(z, s)/\partial s\Big|_{s=a}, \quad \beta_{\lambda_0,n}(z) = \partial B_{\lambda_0,n}(z, s)/\partial s\Big|_{s=b},$$

and reduce to $\alpha_{2,n}(z)$, $\beta_{2,n}(z)$ for $\lambda_0=0$. Now the generating functions in (61) satisfy (in z) the differential equation

$$[D^2-\lambda^2](\quad)=[(D^2-\lambda_0^2)-(\lambda^2-\lambda_0^2)](\quad)=0$$

and take on at $z=a$, $z=b$, the values 1, 0; 0, 1, respectively. Hence,

$$(D^2-\lambda_0^2)\alpha_{\lambda_0,n}(z)=\begin{cases}0 & \text{for } n=0,\\ \alpha_{\lambda_0,n-1} & \text{for } n>0,\end{cases}$$

$$\alpha_{\lambda_0,n}(a)=\begin{cases}1 \text{ for } n=0,\\ 0 \text{ for } n>0,\end{cases}$$

$$\alpha_{\lambda_0,n}(b)=0,$$

and similar relations hold for $\beta_{\lambda_0,n}$. From these facts it follows that $\alpha_{\lambda_0,n}(z)$, $\beta_{\lambda_0,n}(z)$ are solutions of $(D^2-\lambda_0^2)^{n+1}(\quad)=0$ and that their derivatives of order $0, 2, \cdots, 2n$ at $z=a$, $z=b$, agree with the corresponding derivatives of $\alpha_n(z)$, $\beta_n(z)$ at these points.

We may now generalize formula (16) by establishing

$$f(z)-\sum_{i=0}^{n}\left[(D^2-\lambda_0^2)^i f(s)\Big|_{s=a}\alpha_{\lambda_0,i}(z)+(D^2-\lambda_0^2)^i f(s)\Big|_{s=b}\beta_{\lambda_0,i}(z)\right]$$
(62)
$$=\int_z^a A_{\lambda_0,n}(z,s)(D^2-\lambda_0^2)^{n+1}f(s)ds+\int_z^b B_{\lambda_0,n}(z,s)(D^2-\lambda_0^2)^{n+1}f(s)ds.$$

This is done through successive applications of

$$\int_{s_1}^{s_2}[u(s)(D^2-\lambda_0^2)v(s)-v(s)(D^2-\lambda_0^2)u(s)]ds=u(s)v'(s)-u'(s)v'(s)\Big]_{s_1}^{s_2}$$

to the pairs of functions $A_{\lambda_0,0}(z,s)$, $f(s)$; $A_{\lambda_0,1}(z,s)$, $f''(s)$; $\cdots$, and in a manner quite analogous to the way in which (16) was deduced. From the properties of $\alpha_{\lambda_0,i}$, $\beta_{\lambda_0,i}$ it is possible to show that the finite sum on the left of (62)—it might conveniently be denoted by $P_{\lambda_0,2,n+1}(z)$—is a solution of $(D^2-\lambda_0^2)^{n+1}(\quad)=0$ and that its derivatives of order $0, 2, \cdots, 2n$ at $z=a$, $z=b$, agree with those of $f(z)$. The function $P_{\lambda_0,2,n}(z)$ we shall consider as an approximation to $f(z)$; it is of the form $e^{\lambda_0 z}Q_1(z)+e^{-\lambda_0 z}Q_2(z)$, where Q_1, Q_2 are polynomials in z of at most the $(n-1)$th degree.

As an analogue of Theorem 1, one might seek for sufficient conditions in order that $P_{\lambda_0,2,n}$ approach $f(z)$ as n becomes infinite, conditions of the form $f(z)=O(e^{k|z|})$, where k is a properly restricted constant. For such functions the inequality

(24) $$|f^n(z)|<Cn^{1/2}k^n$$

holds. Hence

$$|f^n(z)| < CK^n, \ K < k,$$

with a possibly different C, and

$$|(D^2 - \lambda_0^2)^n f(z)| < C(K^2 + |\lambda_0|^2)^n.$$

Now the functions $A(z, s; \lambda)$, $B(z, s; \lambda)$ are analytic in the λ^2-plane in a circle of radius $|\lambda_0^2 - \lambda_1^2|$ (λ_1^2 is the pole, or either of the two poles, of these functions which is nearest to λ_0^2). Hence,

$$A_{\lambda_0,n}(z, s), \ B_{\lambda_0,n}(z, s) = O[(|\lambda_0^2 - \lambda_1^2| + \epsilon)]^{-n}, \ \epsilon > 0.$$

Combining this with the preceding inequality, we infer that the right-hand member of (62) approaches zero as n becomes infinite if

$$(K^2 + |\lambda_0|^2)/(|\lambda_0^2 - \lambda_1^2| + \epsilon) < 1.$$

From this it appears that a sufficient condition in order that $P_{\lambda_0,2,n}(z)$ approach $f(z)$ is that

$$f(z) = O(e^{k|z|}), \ k^2 < |\lambda_0^2 - \lambda_1^2| - |\lambda_0^2|.$$

The last inequality for k is non-vacuous only if its right-hand member is positive, that is, if the origin is nearer to λ_0^2 than any of the poles of A, B. (When such is the case, λ_1^2 is necessarily the pole $\lambda^2 = -\pi^2/(a-b)^2$.) However, even when it is non-vacuous, the sufficient condition just found is quite inadequate to characterize the functions that may be approximated to an arbitrary degree by means of $P_{\lambda_0,2,n}(z)$. This may be seen by considering the example $f(z) = \sinh k(z-b)$, where k is a constant. It may be shown directly that now the sequence $P_{\lambda_0,2,n}(z)$ converges to $f(z)$ when and only when

$$|k^2 - \lambda_0^2| < |\lambda_1^2 - \lambda_0^2|.$$

The last example shows that conditions of the type $f(z) = O(e^{k|z|})$ are not properly suited for the problem at hand. More effective conditions may be given in form of inequalities involving $(D^2 - \lambda^2)^n f(z)$.

Let now the poles of A, B, arranged in order of non-decreasing distance from λ_0^2, be

$$\lambda_1^2, \lambda_2^2, \cdots,$$

and suppose that $|\lambda_0^2 - \lambda_1^2| < |\lambda_0^2 - \lambda_2^2| < |\lambda_0^2 - \lambda_3^2|$. Each of the poles λ_i^2 is equal to $-k_i^2\pi^2/(a-b)^2$, where k_i is a proper integer; the above inequalities could readily be shown to restrict λ_0^2 from lying on certain parallel straight lines. One may prove that a sufficient as well as essentially necessary condition for the convergence of

$$\sum_{n=0}^{\infty}[C_n\alpha_{\lambda_0,n}(z) + D_n\beta_{\lambda_0,n}(z)]$$

is that the two series

$$\sum_{n=0}^{\infty}[C_n + (-1)^{k_1+1}D_n]/(\lambda_0^2 - \lambda_1^2)^n,$$

$$\sum_{n=0}^{\infty}[C_n + (-1)^{k_1}D_n]/(\lambda_0^2 - \lambda_2^2)^n$$

be convergent; here k_1 is given by

$$k_1^2 = -\lambda_1^2(a-b)^2/\pi^2.$$

In the convergent case, the limit function may be shown to be of exponential type at most equal to $k_2\pi/|a-b|$.

By replacing C_n, D_n above by $(D^2-\lambda_0^2)^n f(z)|_{z=a}$, $(D^2-\lambda_0^2)^n f(z)|_{z=b}$, one may obtain necessary and sufficient conditions for $P_{\lambda_0,2,n}(z)$ to converge. When these are satisfied, one may show that $f(z)$ differs from the limit approached by $P_{\lambda_0,2,n}(z)$ by a linear combination of sin $[k\pi(z-a)/(a-b)]$; $k=1, 2, \cdots, k_2$.

For $\lambda_0=0$, the above results reduce to those of Theorems 6–8.

20. **Expansions of the Green's functions about a pole and the approximations they suggest.** To illustrate the new features that occur when the Green's functions are expanded in a Laurent series in the parameter in the neighborhood of a pole, we shall consider in this section the Green's functions $A(z,s;\lambda)$, $B(z, s; \lambda)$ defined by means of (8), (9_3), (9_4), and the two further boundary conditions

$$\left.\frac{\partial A(z,s;\lambda)}{\partial s}\right|_{s=a} = 0, \quad \left.\frac{\partial B(z,s;\lambda)}{\partial s}\right|_{s=b} = 0.$$

These functions are not to be confused with the Green's functions A, B of Part II. We find

$$(63)\qquad A(z,s;\lambda) = -\frac{\cosh\lambda(z-b)\cosh\lambda(s-a)}{\lambda\sinh\lambda(a-b)},\qquad B(z,s;\lambda) = \frac{\cosh\lambda(z-a)\cosh\lambda(s-b)}{\lambda\sinh\lambda(a-b)}.$$

These functions are connected with the system ($3_2'$), $u'(a)=u'(b)=0$, in the same way that the functions defined by (8), (9_i) are connected with the system ($3_2'$), (4_2). It will be observed, however, that now $\lambda=0$ is a pole of the second order for the functions A, B; this is connected with the fact that a non-trivial solution of the above system exists for $\lambda=0$, namely, $u(z)=$constant.

We shall now denote by $F(z)$ any iterated integral of $f(z)$ (for example, $\int_a^z (z-s)f(s)ds$), expand A, B in powers of λ^2:

$$A(z, s; \lambda) = \sum_{n=-1}^{\infty} A_n(z, s)\lambda^{2n}, \quad B(z, s; \lambda) = \sum_{n=-1}^{\infty} B_n(z, s)\lambda^{2n},$$

and apply the formula (7_2) to the pairs of functions

$$F(z),\ A_{-1}(z, s);\ f(s),\ A_0(z, s);\ \cdots;\ f^{(2n)}(s),\ A_n(z, s)$$

between the limits z and a; and to the functions

$$F(z),\ B_{-1}(z, s);\ f(s),\ B_0(z, s);\ \cdots;\ f^{(2n)}(s),\ B_n(z, s)$$

between the limits z and b, and add the resulting equations. On making use of the various differential and boundary-value properties of $A_i(z, s)$, $B_i(z, s)$ we get

$$\begin{aligned} f(z) = {} & A_{-1}(z, a)F'(a) + B_{-1}(z, b)F'(b) + \sum_{i=0}^{n} [A_i(z, a)f^{(2i+1)}(a) \\ & + B_i(z, b)f^{(2i+1)}(b)] + \int_z^a f^{(2n+2)}(s)A_n(z, s)ds \\ & + \int_z^b f^{(2n+2)}(s)B_n(z, s)ds. \end{aligned} \tag{64}$$

The nature of the above summation is understood from the formulas

$$A_n(z, a) = \alpha'_{n+1}(z),\ B_n(z, b) = \alpha'_{n+1}(z) \quad (n = -1, 0, 1, \cdots),$$

where $\alpha_n(z)$, $\beta_n(z)$ are the polynomials of Part II. These equations are immediately deduced from (63) and (17_i). The summation in (64) may now be easily shown to be a polynomial of degree at most $2n+2$ whose derivatives of order $2i+1$, $i=0, 1, \cdots, n$, at $z=a$, $z=b$ are equal to corresponding derivatives of $f(z)$ at those points. The term preceding the summation in (64) is

$$(F'(b) - F'(a))/(b - a) = \int_a^b f(s)ds/(b - a),$$

and may be ascribed to the pole of the Green's functions at the origin.

A treatment of the approximations suggested in this fashion may, of course, be given along the lines of Part II. The problem could, however, be shown to be equivalent to the approximations by means of $P_{2,n}(z)$ as follows.

Suppose that the function $f'(z)$ can be uniformly approximated by means of the polynomials $P_{2,n}(z)$ so that

$$\left| f'(z) - \sum_{i=0}^{n} [f^{(2i+1)}(a)\alpha_i(z) + f^{(2i+1)}(b)\beta_i(z)] \right| < \epsilon$$

in an arbitrary region enclosing a and b, for n large enough. Integrating the "remainder" and making use of

$$\int \alpha_n(z)dz = \alpha'_{n+1}(z) + C,^*$$

we infer that for proper constants C_n

$$\lim_{n\to\infty} f(z) - \sum_{i=0}^{n} [f^{(2i+1)}(a)\alpha'_{i+1}(z) + f^{(2i+1)}(b)\beta'_{i+1}(z)] - C_n = 0$$

uniformly in an arbitrary given region. Integrating the left hand member from a to b we get

$$\lim_{n\to\infty} \int_a^b f(z)dz - C_n(b-a) = 0.$$

Hence C_n above could be replaced by the constant $\int_a^b f(z)dz/(b-a)$.

21. **Expansions connected with a first-order differential equation and a two-point boundary condition.** We shall be concerned in this section with approximations to $f(z)$ by means of polynomials $P_n(z)$ of degree at most $n-1$, such that

(65) $P_n^{(i)}(a) + kP_n^{(i)}(b) = f^{(i)}(a) + kf^{(i)}(b)$ $(i = 0, 1, \cdots, n-1; n = 1, 2, \cdots)$,

where a and b are two given fixed points, and k is a given constant which we suppose different from 0 or -1.

To prove the existence and uniqueness of the polynomials $P_n(z)$, consider the function

$$C(z, \lambda) = e^{\lambda(z-b)}/(e^{\lambda(a-b)} + k).$$

Since $k \neq 0, -1$, the denominator above vanishes for an infinite number of values of λ, none of which is equal to 0; these roots of the denominator are simple. We may therefore expand $C(z, \lambda)$ in powers of λ:

(66) $$C(z, \lambda) = \sum_{n=0}^{\infty} \lambda^n \alpha_n(z),$$

* See footnote in connection with equations (17_i).

where the expansion converges for $|\lambda| < |\lambda_1|$, λ_1 being the pole or either of the two poles of $C(z, \lambda)$ nearest the origin of the λ-plane. Now $C(z, \lambda)$ satisfies the equations

$$\partial C(z, \lambda)/\partial z = \lambda C(z, z),$$

$$C(a, \lambda) + kC(b, \lambda) = 1.$$

Hence $\alpha_n(z)$ satisfies the conditions

$$\alpha_n'(z) = \begin{cases} 0 & \text{for } n = 0, \\ \alpha_{n-1} & \text{for } n > 0, \end{cases}$$

$$\alpha_n(a) + k\alpha_n(b) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}$$

From this we conclude that $\alpha_n(z)$ is a polynomial of degree n, and that

$$\alpha_n^{(i)}(a) + k\alpha_n^{(i)}(b) = \begin{cases} 0 \text{ for } i \neq n, \\ 1 \text{ for } i = n. \end{cases}$$

It is now obvious that the polynomial

$$\sum_{i=0}^{n} [f^{(i)}(a) + kf^{(i)}(b)]\alpha_i(z)$$

satisfies the conditions (65) postulated for $P_n(z)$. The existence of a polynomial satisfying these conditions is thus proved. The uniqueness of $P_n(z)$ now follows from the fact that a polynomial satisfying conditions (65) exists for arbitrary values of the right-hand members of (65).

To discuss the convergence of $P_n(z)$ to $f(z)$, we introduce two Green's functions $A(z, s; \lambda)$, $B(z, s; \lambda)$:

$$(67) \qquad \begin{aligned} A(z, s; \lambda) &= e^{\lambda(z-s-b)}/(e^{-\lambda b} + ke^{-\lambda a}), \\ B(z, s; \lambda) &= ke^{\lambda(z-s-b)}/(e^{-\lambda b} + ke^{-\lambda a}). \end{aligned}$$

These functions obviously satisfy the equations

$$(68) \qquad \partial A(z, s; \lambda)/\partial s = -\lambda A(z, s; \lambda), \quad \partial B(z, s; \lambda)/\partial s = -\lambda B(z, s; \lambda),$$

$$(69) \qquad \begin{aligned} A(z, z; \lambda) + B(z, z; \lambda) &= 1, \\ -kA(z, a; \lambda) + B(z, b; \lambda) &= 0, \end{aligned}$$

and possess the same poles as the function $C(z, \lambda)$. In fact

$$C(z, \lambda) = A(z, a; \lambda) = B(z, b; \lambda)/k.$$

The expansions of A, B in powers of λ,

$$(70)\qquad \begin{aligned} A(z, s; \lambda) &= \sum_{n=0}^{\infty} A_n(z, s)\lambda^n, \\ B(z, s; \lambda) &= \sum_{n=0}^{\infty} B_n(z, s)\lambda^n \end{aligned}$$

are valid within the same circle in the λ-plane as (66).

We next apply

$$(7_1)\qquad \int_{s_1}^{s_2} [u(s)v'(s) + u'(s)v(s)]ds = u(s)v(s)\Big]_{s_1}^{s_2}$$

to the pairs of functions $A_0(z, s)$, $f(s)$; $A_1(z, s)$, $f'(s)$; $\cdots$; $A_n(z, s)$, $f^{(n)}(s)$, between the limits z and a; then to the pairs of functions $B_0(z, s)$, $f(s)$; $\cdots$; $B_n(z, s)$, $f^{(n)}(s)$, between the limits z and b. Adding the resulting equations, and making use of the various properties of A_i, B_i, that result when the series (70) are substituted in (68) and (69), and the coefficients of like powers of λ equated on both sides, we obtain in a familiar manner the formula

$$\begin{aligned} f(z) = &\sum_{i=0}^{n} [f^{(i)}(a) + kf^{(i)}(b)]A_i(z, a) \\ &+ \int_z^a A_n(z, s)u^{(n+1)}(s)ds + \int_z^b B_n(z, s)u^{(n+1)}(s)ds, \end{aligned}$$

or the "remainder" formula

$$f(z) - P_{n+1}(z) = \int_z^a A_n(z, s)u^{(n+1)}(s)ds + \int_z^b B_n(z, s)u^{(n+1)}(s)ds.$$

Using the last form of the remainder, there is no difficulty in showing that a sufficient condition for the convergence of $P_n(z)$ to $f(z)$ is that, in addition to its being an integral function, $f(z)$ be of exponential type less than $|\lambda_1|$.

As regards necessary conditions for the convergence of $P_n(z)$, the results are even simpler than for the polynomials discussed in Part II. We shall first suppose that k is not a real positive number. There will then exist one pole of A, B, C, namely λ_1, which is nearest the origin. Equation (66) now leads to the asymptotic representation

$$\alpha_n(z) = \text{const. } e^{\lambda_1 z}/\lambda_1^n + O[(|\lambda_2| - \epsilon)^{-n}], \ \epsilon > 0,$$

where λ_2 is the pole next nearest to the origin.

By means of this asymptotic representation one may study the convergence of the series $\sum C_n\alpha_n(z)$, and prove that this series either converges for all z, or diverges for all z, depending upon whether the series $\sum C_n/\lambda_1^n$ converges or not. In the former case, the sum of $\sum C_n\alpha_n(z)$ is a function of exponential type at most equal to $|\lambda_1|$.

Applying these results to the case where $C_n=f^{(n)}(a)+kf^{(n)}(b)$ one obtains necessary conditions in order that $P_n(z)$ converge to $f(z)$. When these conditions are satisfied, the limit function of $P_n(z)$, $l(z)$, has the property that

$$f^{(n)}(a) + kf^{(n)}(b) = l^{(n)}(a) + kl^{(n)}(b) \qquad (n = 0, 1, \cdots).$$

Hence $f(z)-l(z)$ is a solution of the difference equation

$$g(z) + kg(z + b - a) = 0.$$

Now any solution of this difference equation is of the form $e^{\lambda_1 z}p(z)$, where $p(z)$ is periodic of period $b-a$. Combining this fact with the fact that $f(z)$, $l(z)$ are of exponential type at most equal to $|\lambda_1|$, one may prove that the difference $l(z)-f(z)$ is representable by a finite Fourier series.

Suppose next that k is real and positive, so that there are two roots of $e^{\lambda(a-b)}+k$ that are nearest the origin, namely,

$$(\log k \pm \pi i)/(a - b), \log k \text{ real};$$

denote them by λ_1, λ_2 respectively ($|\lambda_1| = |\lambda_2|$). We now have two poles on the circle of convergence of the expansions of the various generating functions in powers of the parameter, a situation which either has not presented itself hitherto or has been artificially excluded. The previous asymptotic representation of $\alpha_n(z)$ now has to be replaced by

$$\alpha_n(z) = \frac{e^{\lambda_1(z-b)}}{k_1(b - a)\lambda_1^n} + \frac{e^{\lambda_2(z-b)}}{k_1(b - a)\lambda_2^n} + O[(|\lambda_3| + \epsilon)^{-n}], \epsilon > 0.$$

By using it, it is possible to show that in order that $\sum C_n\alpha_n(z)$ converge for two arbitrary values of z it is necessary that both series

$$\sum C_n/\lambda_1^n, \quad \sum C_n/\lambda_2^n$$

involving the *same* constants C_n, converge. Conversely, when both of these series converge, $\sum C_n\alpha_n(z)$ converge for all z. In other respects this case does not differ from the preceding case with a single pole on the circle of convergence.

22. **Certain boundary value expansions of functions of several variables.** As stated in §4, the expansions which we shall consider in this section formed the starting point of the investigation that resulted in the present paper.

Let R be an arbitrary finite region, for definiteness in real euclidean space of three dimensions, S its bounding surface, and let $f(x, y, z)$ be a function analytic within and on S. We shall suppose that S is sufficiently regular so that the Dirichlet problem for its interior, R, has a unique solution. We shall consider the question of approximating to f by means of the sequence of functions $p_n(x, y, z)$, $n=1, 2, \cdots$, where p_n is determined by means of the equations

$$(71) \qquad \nabla^{2n} p_n = 0,$$

$$(72) \qquad p_n = f,\ \nabla^2 p_n = \nabla^2 f,\ \cdots,\ \nabla^{2n-2} p_n = \nabla^{2n-2} f \text{ over } S.$$

To prove the existence of the approximations in question and to obtain a formula for the remainder $f-p_n$, we shall introduce a sequence of functions G_0, $G_1, \cdots$ defined as follows: the first member of the sequence, $G_0=G_0(P, P')$, is *the* Green's function of potential theory for the region R, that is, it is a function harmonic in the coördinates of P inside R, except for P at P', where G_0 plus the reciprocal of the distance from P' is harmonic, and it vanishes on S, the boundary of R; the succeeding members of the sequence, $G_i=G_i(P, P')$ for $i>0$, are defined by means of

$$(73) \qquad \nabla_P^2 G_i = G_{i-1},$$

$$(74) \qquad G_i(P, P') = 0 \text{ for } P \text{ on } S.$$

The solution G_i of (73), (74) may be expressed in terms of G_0 and G_{i-1} by means of a familiar integral form. These integrals are improper but convergent, and represent functions analytic for both point arguments P, P' inside S except for P and P' coincident. Thus, the integrand leading to G_1 becomes infinite when the point of integration approaches P or P' like the negative reciprocal of the distance from that point, and therefore the integral is convergent. The singularities of $G_i(P, P')$ for coincident P, P' get successively milder (as judged from the point of view of functions of a real variable) with increasing i. For, any solution u of the equation

$$\nabla^2 u = \text{an analytic function}$$

is also analytic. Therefore, and since G_0+r^{-1} is analytic without exception for P inside R, it follows by induction that for any i, $G_i+r^{2i-1}/(2i)!$ is analytic for P inside R; hence the singularity of G_i at $P=P'$ (that is, for P coincident with P') is precisely the same as that of $-r^{2i-1}/(2i)!$. From this we conclude that while G_i is non-analytic for P at P', it is of class $C^{(2i-2)}$ there.

From the integral expression of G_i, $i>0$, in terms of G_0 it follows that the functions G_n form the iterated kernels of the kernel G_0 of the integral equation

$$(75) \qquad u(P) = v(P) + \lambda^2 \int_R G_0(P, P')u(P')dP',$$

where the integration extends over R; this integral equation is equivalent to the differential equation

$$\nabla^2 u - \lambda^2 u = \nabla^2 v$$

and to the boundary condition

$$u = 0 \text{ on } S.$$

The kernel of the integral equation (75) becomes infinite for $P = P'$, but it is well known that, except for a countable set of real "characteristic" values of λ^2, (75) possesses a unique solution for an arbitrary v, while for each characteristic value of λ^2 the homogeneous integral equation obtained by putting $v \equiv 0$ possesses a non-trivial solution (representing a mode of free vibration of the cavity inside S). The theory of the solutions of the equivalent differential system, in fact, antedates the Fredholm theory, and served as one of the landmarks in the development of the latter. With proper modifications, the Fredholm theory may be applied, and the solution of (74) expressed by means of a resolvent, whose poles are the above characteristic parameter values,* and the Schmidt theory invoked to prove the existence and the reality of the characteristic values. The functions G_n are the coefficients which result when the resolvent is expanded in powers of λ^2.

One way of applying the Fredholm theory to (75), due to Fredholm himself, is to replace $u(P')$ in the integrand by the value obtained from the right-hand member; thereupon the integral equation is changed into one with the finite kernel G_1; the resolvent of the original integral equation may be simply expressed in terms of the resolvent of the resulting equation.† From this it is seen that for $n > 0$, the G_n satisfy an inequality of the form

$$|G_n(P, P')| < C(\rho^2 + \epsilon)^{-2n}, \quad \epsilon > 0, \tag{76}$$

where ρ^2 is the smallest characteristic value of the parameter λ^2 of (75), and C is a constant independent of P and P'.‡

We now apply Green's theorem

$$\int (U\nabla^2 V - V\nabla^2 U)dP = \int [U(\partial V/\partial n) - V(\partial U/\partial n)]dS$$

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* For references to the literature, see Hellinger-Toeplitz, Encyklopädie der Mathematischen Wissenschaften, II C 13, 12, 13 (a).

† Hellinger-Toeplitz, loc. cit., 13 (b).

‡ An inequality of this type follows for $n > 2$ without the use of the theory of integral equations from the fact that G_1, G_2 are bounded, and by the use of $G_n = \int G_1 G_{n-2} dP$ for $n > 2$. More precise asymptotic estimates may be developed for G_n, of a nature similar to (20) in the one-dimensional case.

to the functions $f(P)$, $G_1(P, P')$ (for a fixed P') over the region inside S and outside a small sphere whose center is at P'; also to each pair of functions $\nabla^2 f(P)$, $G_1(P, P')$; $\nabla^4 f(P)$, $G_2(P, P')$; $\cdots$; $\nabla^{2n} f(P)$, $G_n(P, P')$ over the same region. Adding the resulting equations, and making use of (73), we find that the sum of the volume integrands reduces to

$$- G_n(P, P')\nabla^{2n+2} f(P).$$

If we now let the small sphere shrink down to the point P', we see from the analyticity of the functions $G_i + [r^{2i-1}/(2i)!]$, that all the surface integrals over the sphere approach zero with the exception of $\int(\partial G_0/\partial n)fdS$, which (as is well known) approaches $4\pi f(P')$. As regards the surface integrals over S, one-half of them reduce to zero on account of (74). We thus get the formula

$$\begin{aligned} 4\pi f(P') = & \sum_{i=0}^{n} \int_S \nabla^{2i} f(P_s)(\partial G_i(P_s, P')/\partial n) dS \\ & + \int_R G_n(P, P')\nabla^{2n+2} f(P) dP. \end{aligned} \tag{77}$$

As is known, if S is sufficiently regular, so that the Dirichlet problem for its interior R has a solution, there exists a function $u(P')$ of class C'' inside S, continuous with its first derivatives at S, vanishing on S, and satisfying in R the differential equation

$$\nabla^2 u(P') = v(P'),$$

where v is an arbitrary continuous function. For S so restricted one may prove by induction the existence and uniqueness of a function $u_n(P')$ of class $C^{(2n+2)}$ in R, $C^{(2n+1)}$ at S, and such that

$$\nabla^{2n+2} u_n(P') = v(P') \text{ in } R, \tag{78}$$

$$u_n(P') = \nabla^2 u_n(P') = \cdots = \nabla^{2n} u(P') = 0 \text{ on } S. \tag{79}$$

Suppose then that we put this function $u_n(P')$ in place of $f(P')$ in (77); all the surface integrals vanish, and we get

$$\begin{aligned} 4\pi u_n(P') &= \int_R G_n(P, P')\nabla^{2n+2} f(P) dP \\ &= \int_R G_n(P, P') v(P) dP. \end{aligned}$$

Hence we conclude that the volume integral in (77) represents the function that is determined by means of (78), (79) when v is replaced by $\Delta^{2n+2} f$. Consequently the sum of the surface integrals in (77) represents a function p_n satisfying the conditions (71), (72). The uniqueness of p_n also follows from (77), provided that the uniqueness of the solution of (78), (79) is kept in mind.

If in (77) we let n become infinite, we are led to consider the infinite series representation (80) below; with regard to the validity of this representation, we state

THEOREM 13. *In order that*

$$4\pi f(P') = \sum_{n=0}^{\infty} \int_S \nabla^{2n} f(P_s)(\partial G_n(P_s, P')/\partial n)dS \tag{80}$$

hold for P inside R, it is sufficient that the analytic function f be dominated by a function

$$Ce^{c_1x+c_2y+c_3z},$$

where C, c_i are (positive) constants, and

$$c_1^2 + c_2^2 + c_3^2 < \rho^2,$$

ρ^2 being the smallest value of λ^2 for which there exists in R a function $u \neq 0$, vanishing on the boundary S, and satisfying

$$\nabla^2 u - \lambda^2 u = 0$$

in R.

The proof of this theorem follows readily by applying (76) as well as the inequality

$$| \nabla^{2n} u | < C' \rho^{2n}$$

to the volume integral in (77) (the latter of the above inequalities holds for a proper constant C' for (x, y, z) in R). The volume integral is thus seen to converge to zero uniformly over R. Moreover, under the conditions stated, it may be shown that if the order of summation and integration in (80) be interchanged, the resulting summation in the integrand converges uniformly for P' in R and P_s over S; hence the integration may be carried out after the summation.

A boundary value expression analogous to (80) may be established for an arbitrary number of dimensions n. The singularity of the Green's functions has to be properly modified, and the constant 4π in the left hand member of (80) has to be replaced by the $(n-1)$-content of a unit $(n-1)$-sphere. It is found that an increasingly large number of members of the sequence G_0, G_1, $\cdots$ fail to remain bounded.* As a result it is found that the details of

* The functions taking the place of $r^{2i-1}/(2i)!$ in displaying the nature of the singularity of G_n at coincident P and P' are discussed in the author's paper *On certain integrals over spheres*, reported to the Society in December, 1928.

applying the Fredholm theory have to be modified with n. In the treatment of the latter polynomials, it will be recalled that the independent variable was allowed to range over the complex plane. Now a similar boundary value expression to the one considered could probably be given for the region outside a surface S, but it would be of decided interest to generalize the formulas in question to the complex domain, where, even for a real surface S, the distinction between the inside and the outside of S would dissolve, after the same manner that the inside and outside of an interval get connected when the interval is immersed in the complex plane.

For $n=1$ the region R reduces to an interval (a, b), and the approximations p_n become the polynomials $P_{2,n}$ of Part II.*

* It has been pointed out to the author that the results concerning the existence of roots of certain exponential sums which have been above attributed to Pólya (see footnote, p. 303) had been obtained at an earlier date by J. D. Tamarkin. For reference to the latter's treatment, as well as for more complete discussion of zeros of exponential sums, see R. E. Langer, Bulletin of the American Mathematical Society, April, 1931, pp. 213–239.

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